

ON THE MICROLOCAL PROPERTIES OF THE RANGE OF SYSTEMS OF PRINCIPAL TYPE

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ABSTRACT. The purpose of this paper is to study microlocal conditions for inclusion relations between the ranges of square systems of pseudodifferential operators which fail to be locally solvable. The work is an extension of earlier results for the scalar case in this direction, where analogues of results by L. Hörmander about inclusion relations between the ranges of first order differential operators with coefficients in C^∞ which fail to be locally solvable were obtained. We shall study the properties of the range of systems of principal type with constant characteristics for which condition (Ψ) is known to be equivalent to microlocal solvability.

1. INTRODUCTION

In this paper we shall study the properties of the range of a square system of classical pseudodifferential operators $P \in \Psi_{\text{cl}}^m(X)$ on a C^∞ manifold X of dimension n , acting on distributions $\mathcal{D}'(X, \mathbb{C}^N)$ with values in \mathbb{C}^N ; if $u \in \mathcal{D}'(X, \mathbb{C}^N)$ then $u = (u_j)_{j=1, \dots, N}$ where $u_j \in \mathcal{D}'(X)$. If $P = (P_{jk})$ is an $N \times N$ system, then $Pu \in \mathcal{D}'(X, \mathbb{C}^N)$ is defined by

$$(1.1) \quad (Pu)_j = \sum_{k=1}^N P_{jk} u_k, \quad 1 \leq j \leq N.$$

Here classical means that the symbol of P is an asymptotic sum $P_m + P_{m-1} + \dots$ of matrix valued smooth functions where $P_j(x, \xi)$ is homogeneous of degree j in ξ , and P_m is the principal symbol.

We shall restrict our study to systems of principal type, which means that the principal symbol vanishes of first order on the kernel, see Definition 2.1. We shall also assume that all (systems of) operators are properly supported, that is, both projections from the support of the operator kernel in $X \times X$ to X are proper maps. For such $N \times N$ systems, local solvability at a compact set $M \subset X$ means that for every f in a subspace of $C^\infty(X, \mathbb{C}^N)$ of finite codimension the equation

$$(1.2) \quad Pu = f$$

has a local weak solution $u \in \mathcal{D}'(X, \mathbb{C}^N)$ in a neighborhood of M . We can also define microlocal solvability at a set in the cosphere bundle, or equivalently, at a conic set in $T^*(X) \setminus 0$, the cotangent bundle of X with the zero section removed. By a conic set $K \subset T^*(X) \setminus 0$ we mean a set that is conic in the fiber, that is,

$$(x, \xi) \in K \implies (x, \lambda \xi) \in K \quad \text{for all } \lambda > 0.$$

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If, in addition, $\pi_x(K)$ is compact in X , where $\pi_x : T^*(X) \rightarrow X$ is the projection, then K is said to be compactly based. Thus, we say that P is solvable at the compactly based cone $K \subset T^*(X) \setminus 0$ if there is an integer N_0 such that for every $f \in H_{(N_0)}^{\text{loc}}(X, \mathbb{C}^N)$ there exists a $u \in \mathcal{D}'(X, \mathbb{C}^N)$ with $K \cap WF(Pu - f) = \emptyset$ (see Definition 4.1).

The famous example due to Hans Lewy [13] showed that not all smooth linear differential operators are solvable. This example led to an extension due to Hörmander [6, 7] in the sense of a necessary condition for a differential equation $P(x, D)u = f$ to have a solution locally for every $f \in C^\infty$. In fact (see [8, Theorem 6.1.1]), if Ω is an open set in \mathbb{R}^n , and P is a differential operator of order m with coefficients in $C^\infty(\Omega)$ such that the differential equation $P(x, D)u = f$ has a solution $u \in \mathcal{D}'(\Omega)$ for every $f \in C_0^\infty(\Omega)$, then $\{p, \bar{p}\}$ must vanish at every point $(x, \xi) \in \Omega \times \mathbb{R}^n$ for which $p(x, \xi) = 0$, where p is the principal symbol of P and

$$\{a, b\} = \sum_{j=1}^n \partial_{\xi_j} a \partial_{x_j} b - \partial_{x_j} a \partial_{\xi_j} b$$

denotes the Poisson bracket.

Recall that a scalar pseudodifferential operator P is of principal type if the Hamilton vector field H_p of the principal symbol p is not proportional to the radial vector field ρ when $p = 0$, where $H_p : f \mapsto \{p, f\}$ for $f \in C^\infty$ and ρ is given in terms of local coordinates on $T^*(X) \setminus 0$ by $\xi \partial_\xi$. For such operators it was conjectured by Nirenberg and Treves [15] that local solvability at a compact set $M \subset X$ in the sense of (1.2) is equivalent to condition (Ψ) on the principal symbol, which means that there is a neighborhood Y of M such that

$$(1.3) \quad \text{Im } ap \text{ does not change sign from } - \text{ to } +$$

along the oriented bicharacteristics of $\text{Re } ap$

over Y for any $0 \neq a \in C^\infty(T^*(Y) \setminus 0)$. The oriented bicharacteristics of $\text{Re } ap$ are the positive flow-outs of the Hamilton vector field $H_{\text{Re } ap}$ on $\text{Re } ap = 0$, sometimes referred to as semi-bicharacteristics of p . Note that condition (1.3) is invariant under multiplication of p with non-vanishing factors and symplectic changes of coordinates. Hence the condition is invariant under conjugation of P with elliptic Fourier integral operators.

The necessity of condition (Ψ) for local solvability of scalar pseudodifferential operators of principal type was proved by Moyer [14] in 1978 for the two dimensional case and by Hörmander [9] in 1981 for the general case. It was finally shown by the first author [2] in 2006 that condition (Ψ) is also sufficient for local and microlocal solvability for scalar operators of principal type.

For systems, no corresponding conjecture for solvability exists. However, by considering the case when the principal symbol of a square system P of principal type has *constant characteristics* (see Definition 2.2), the first author [4] showed that local and microlocal solvability is equivalent to condition (Ψ) on the eigenvalues of the principal symbol. Here we wish to mention that although not explicitly addressed in [4], one actually finds that for systems of principal type with constant characteristics, condition (Ψ) on the eigenvalues of the principal symbol is necessary also for semi-global solvability in the sense of [11, Theorem 26.4.7]. For easy reference we have included a statement of this result, see Theorem 4.4 below and also the reformulation of the result given in Corollary 4.5.

To address a conjecture made by Lewy stipulating that scalar differential operators which fail to have local solutions are essentially uniquely determined by the range, Hörmander [8, Chapter 6.2] proved that if P and Q are two first order differential operators with coefficients in $C^\infty(\Omega)$ and in $C^1(\Omega)$, respectively, such that the equation $P(x, D)u = Q(x, D)f$ has a solution $u \in \mathcal{D}'(\Omega)$ for every $f \in C_0^\infty(\Omega)$, and x is a point in Ω such that

$$(1.4) \quad p(x, \xi) = 0, \quad \{p, \bar{p}\}(x, \xi) \neq 0$$

for some $\xi \in \mathbb{R}^n$, then there is a constant μ such that (at the fixed point x)

$$Q(x, D) = P(x, D)\mu.$$

This result was generalized to scalar classical pseudodifferential operators of principal type by the second author, see [16, Theorem 2.19]. It was shown that if the principal symbol p of $P \in \Psi_{\text{cl}}^m(X)$ fails to satisfy condition (Ψ) along a curve γ in place of the condition given by (1.4), and if the range of $Q \in \Psi_{\text{cl}}^k(X)$ is microlocally contained in the range of P at a cone K containing γ , then one can find an operator $E \in \Psi_{\text{cl}}^{k-m}(X)$ such that all the terms in the asymptotic expansion of the symbol of $Q - PE$ vanish of infinite order at every point belonging to a *minimal bicharacteristic* $\Gamma \subset \gamma$ of p . For the definition of Γ , see Section 3 and Definition 3.3 in particular. It was also shown that one recovers the mentioned result for first order differential operators, if Q is assumed to have C^∞ coefficients. The main result of this paper is a generalization of [16, Theorem 2.19] to systems of principal type with constant characteristics, see Theorem 4.6. We shall only consider operators acting on distributions $\mathcal{D}'(X, \mathbb{C}^N)$ with values in \mathbb{C}^N but since the results are essentially local (see the trivialization given by Proposition 5.1) and invariant under base changes, they immediately carry over to operators on sections of vector bundles.

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2. SYSTEMS OF PRINCIPAL TYPE AND CONSTANT CHARACTERISTICS

Let X be a C^∞ manifold of dimension n . In what follows, C will be taken to be a new constant every time unless stated otherwise. We let $\text{Ker } A$ denote the kernel and $\text{Ran } A$ the range of the matrix A , and let $\mathcal{L}_N = \mathcal{L}(\mathbb{C}^N, \mathbb{C}^N)$ be the space of bounded linear maps from \mathbb{C}^N to \mathbb{C}^N .

In this section we will introduce the systems that will be the focus of our study. For a more thorough discussion as well as multiple examples, we refer to [4]. We begin by recalling the definition of a square system of principal type.

Definition 2.1. We say that the $N \times N$ system $w \mapsto P(w) \in C^1(T^*(X) \setminus 0)$ is of principal type at w_0 if

$$(2.1) \quad \partial_\nu P(w_0) : \text{Ker } P(w_0) \longrightarrow \text{Coker } P(w_0) = \mathbb{C}^N / \text{Ran } P(w_0)$$

is bijective for some $\partial_\nu \in T_{w_0}(T^*(X) \setminus 0)$, where $\partial_\nu P(w_0) = \langle \nu, dP(w_0) \rangle$ and the mapping is given by $u \mapsto \partial_\nu P(w_0)u \bmod \text{Ran } P(w_0)$. We say that $P \in \Psi_{\text{cl}}^m(X)$ is of principal type at w_0 if the principal symbol $P_m(w)$ is of principal type at w_0 .

Due to the relation between the dimensions of the kernel and the cokernel only square systems can be of principal type. Moreover, $P(w) \in C^1$ is of principal type if and only if the adjoint P^* is of principal type, and if $A(w), B(w) \in C^1$ are invertible and $P(w) \in C^1$ is of principal type then APB is of principal type, see [4, Remark 2.2].

Recall that if

$$(2.2) \quad M : \mathbb{R}^+ \times T^*(X) \setminus 0 \rightarrow T^*(X) \setminus 0$$

is the C^∞ map acting through multiplication by t in the fiber, then the radial vector field $\rho \in T(T^*(X) \setminus 0)$ is invariantly described by

$$\rho f = \frac{d}{dt} M_t^* f|_{t=1}, \quad f \in C^1(T^*(X) \setminus 0).$$

Here $M_t(w) = M(t, w)$ and in terms of local coordinates we have $M_t(w) = (x, t\xi)$ and $\rho(w) = \xi \partial_\xi$ at w if $w = (x, \xi)$, see the discussion following [10, Definition 21.1.8]. Suppose now that P is an $N \times N$ system of principal type at w_0 such that Definition 2.1 is satisfied for some $\partial_\nu \in T_{w_0}(T^*(X) \setminus 0)$. If P is homogeneous of degree m , that is, $M_t^* P(w) = t^m P(w)$, then ∂_ν cannot be proportional to $\rho(w_0)$. Indeed, differentiation gives $\rho P = mP$ in view of Euler's homogeneity relation, so $u \mapsto \rho P(w_0)u = 0$ for all $u \in \text{Ker } P(w_0)$. Hence $\rho P(w_0) : \text{Ker } P(w_0) \rightarrow \text{Coker } P(w_0)$ cannot be invertible unless $\text{Ker } P(w_0)$ is trivial.

Remark. For a scalar operator P , Definition 2.1 coincides with the notion that the principal symbol p of P vanishes of first order on the kernel, that is, the differential dp of the principal symbol is non-vanishing at the points where $p = 0$. However, in the homogeneous case one often defines principal type operators so that the Hamilton vector field H_p of p is not proportional to the radial vector field ρ . This is also the definition we shall use for scalar operators of principal type. Although not apparent from Definition 2.1, this would not be an inconvenience due to the properties of minimal bicharacteristics, near which we will do our analysis. However, we would like to point out that if ω is the canonical one form then we recover the scalar definition of principal type from Definition 2.1 applied to scalar symbols under the additional condition that the tangent vector ∂_ν for which the map (2.1) is invertible also satisfies $\langle \partial_\nu, \omega(w_0) \rangle = 0$. In fact, since H_p is proportional to ρ if and only if dp is proportional to ω , the claim follows. Note that if σ is the symplectic form then $\langle \rho, \omega \rangle = \sigma(\rho, \rho) = 0$, so this does not exclude multiples of ρ for which we know that Definition 2.1 does not hold in the homogeneous case in view of the discussion preceding the remark.

The eigenvalues of the principal symbol P_m of an $N \times N$ system of classical pseudodifferential operators $P \in \Psi_{\text{cl}}^m(X)$ are the solutions to the characteristic equation

$$(2.3) \quad |P_m(w) - \lambda \text{Id}_N| = 0,$$

where $|A|$ denotes the determinant of the matrix A . Recall that the algebraic multiplicity of the eigenvalue λ of $P_m(w)$ is the multiplicity of λ as a root to equation (2.3), while the geometric multiplicity is the dimension of $\text{Ker}(P_m(w) - \lambda \text{Id}_N)$. If the matrix $P_m(w)$ depends continuously on a parameter w , then the eigenvalues $\lambda(w)$ also depend continuously on w . Following the terminology in [4], such a continuous function $w \mapsto \lambda(w)$ of eigenvalues will be referred to as a section of eigenvalues of $P_m(w)$. We shall usually only write $\lambda(w)$ to signify this property.

One problem with studying systems $P(w)$ is that the eigenvalues are not very regular in the parameter w , generally they depend only continuously (and eigenvectors measurably) on w , see for example [4, Example 2.16]. We will avoid this problem by studying systems with constant characteristics. Before defining this property we need to introduce some notation.

For an $N \times N$ system $P \in C^\infty(T^*(X) \setminus 0)$ and all integers $k \geq 1$ we define

$$\begin{aligned}\omega_k(P) &= \{(w, \lambda) \in T^*(X) \times \mathbb{C} : \dim \text{Ker}(P(w) - \lambda \text{Id}_N) \geq k\}, \\ \Omega_k(P) &= \{(w, \lambda) \in T^*(X) \times \mathbb{C} : \partial_\lambda^j |P(w) - \lambda \text{Id}_N| = 0 \text{ for all } j < k\}.\end{aligned}$$

Note that $\omega_k(P) = \Omega_k(P) = \emptyset$ for all $k > N$ when P is an $N \times N$ system. We have $\omega_1(P) = \Omega_1(P)$ but $\omega_k(P)$ and $\Omega_k(P)$ could be different when $k > 1$ if P is not symmetric. Clearly, $\omega_k(P)$ and $\Omega_k(P)$ are closed sets for any $k \geq 1$, and

$$\omega_{k+1}(P) \subset \omega_k(P) \subset \Omega_k(P) \subset \Omega_{k-1}(P) \subset \Omega_1(P), \quad k > 1.$$

Therefore, we can define

$$(2.4) \quad \Upsilon(P) = \bigcup_{k \geq 1} \partial \Omega_k(P), \quad \Xi(P) = \bigcup_{k \geq 1} \partial \omega_k(P) \cup \partial \Omega_k(P),$$

where $\partial \omega_k(P)$ and $\partial \Omega_k(P)$ are the boundaries in the relative topology of $\Omega_1(P)$. By the definition we find that the multiplicity of the zeros of $|P(w) - \lambda \text{Id}_N|$ is locally constant on $\Omega_1(P) \setminus \Upsilon(P)$ and the dimension $\dim \text{Ker}(P(w) - \lambda \text{Id}_N)$ is constant on $\Omega_1(P) \setminus (\Xi(P) \setminus \Upsilon(P))$. Thus, we find that both the algebraic and the geometric multiplicities of the eigenvalues of the system $P(w)$ are locally constant on $\Omega_1(P) \setminus \Xi(P)$. Note also that $\Xi(P)$ and $\Upsilon(P)$ are closed and nowhere dense in $\Omega_1(P)$ since they are unions of boundaries of closed sets. Moreover,

$$(w, \lambda) \in \Xi(P) \iff (w, \bar{\lambda}) \in \Xi(P^*)$$

since $P^* - \bar{\lambda} \text{Id}_N = (P - \lambda \text{Id}_N)^*$.

Definition 2.2. We say that the $N \times N$ system $P(w)$ has constant characteristics near the set K if

$$K \times \{0\} \cap \Xi(P) = \emptyset.$$

If K is a compact set, this means that one can find a neighborhood U of K and an $\varepsilon > 0$ so that $U \times D_\varepsilon(0) \cap \Xi(P) = \emptyset$, where $D_\varepsilon(0)$ is the disc at 0 with radius ε .

This is a local definition: if the system has constant characteristics near all points in K , then it has constant characteristics near K . Note also that if K is compact and $K \times \{0\} \cap \Upsilon(P) = \emptyset$, then one can find U and ε as in Definition 2.2 such that $U \times D_\varepsilon(0) \cap \Upsilon(P) = \emptyset$. If $\lambda(w)$ is a section of eigenvalues of $P(w)$ such that $|\lambda(w)| < \varepsilon$ in U then it is a uniquely defined C^∞ function there in view of [4, Remark 2.4]. In particular, if K belongs to the characteristic set $\Sigma(P) = \{w : |P(w)| = 0\}$ of $P(w)$, and $\lambda(w)$ is the section of eigenvalues of $P(w)$ vanishing on K , then after possibly shrinking U we find that $\lambda(w)$ has constant algebraic multiplicity in U , so $\lambda(w) \in C^\infty(U)$ is uniquely defined.

When the principal symbol $P_m(w)$ of an $N \times N$ system of classical pseudodifferential operators $P \in \Psi_{\text{cl}}^m(X)$ is homogeneous of degree m , then the sections of eigenvalues of $P_m(w)$ are also homogeneous of degree m .

Proposition 2.3. Let X be a C^∞ manifold and let $P \in C^\infty(T^*(X) \setminus 0)$ be an $N \times N$ system, homogeneous of degree m , that is, $M_t^* P = t^m P$ where M is the C^∞

map given by (2.2) acting through multiplication by t in the fiber. Then the solutions to the characteristic equation $|P(w) - \lambda \text{Id}_N| = 0$ are continuous and homogeneous of degree m . Furthermore, the number of distinct solutions to

$$(2.5) \quad |P(M_t(w)) - \lambda \text{Id}_N| = 0$$

is a constant function of t .

Proof. Let $w \mapsto \lambda(w)$ be a solution to $|P(w) - \lambda \text{Id}_N| = 0$. Since $P \in C^\infty$ it follows that $\lambda(w)$ is continuous so we only have to prove homogeneity. To this end, introduce a Riemannian metric on X (which by duality allows us to define the unit cotangent bundle), and write $P(x, \xi) = |\xi|^m p(x, \xi)$ where $p(x, \xi) = P(x, \xi/|\xi|)$ is smooth and homogeneous of degree 0. Such functions can be identified with smooth functions on $S^*(X)$, so if $\pi : T^*(X) \setminus 0 \rightarrow S^*(X)$ is the projection then we have $p = \pi^* p_s$ for some matrix valued function $p_s \in C^\infty(S^*(X), \mathcal{L}_N)$. Here p_s depends implicitly on the choice of metric, but this is of no importance. For a fixed point w , suppose that $\varrho_1, \dots, \varrho_\ell$ are the distinct solutions to $|p_s(\pi(w)) - \varrho \text{Id}_N| = 0$. By the homogeneity of P it follows that if $w = (x, \xi)$ in local coordinates, then for any $t > 0$ we have

$$0 = |M_t^* P(w) - M_t^* \lambda(w) \text{Id}_N| = (t|\xi|)^{mN} |p_s(\pi(w)) - (t|\xi|)^{-m} M_t^* \lambda(w) \text{Id}_N|,$$

so there exists an integer $k(t) \in \{1, \dots, \ell\}$ such that

$$(2.6) \quad M_t^* \lambda(w) = (t|\xi|)^m \varrho_{k(t)}.$$

Since $\lambda(w)$ is continuous and the eigenvalues ϱ_k are distinct, equation (2.6) implies that the integer valued map $t \mapsto k(t)$ is locally constant. Since \mathbb{R}^+ is connected, it follows that $k(t) \equiv k$ for some $1 \leq k \leq \ell$. In particular, $k(t) = k(1)$ for all $t > 0$, which yields

$$M_t^* \lambda(w) = (t|\xi|)^m \varrho_{k(t)} = t^m (|\xi|^m \varrho_{k(1)}) = t^m M_1^* \lambda(w) = t^m \lambda(w),$$

so $\lambda(w)$ is homogeneous of degree m .

To prove the last statement of the proposition, let $\ell(t)$ be the number of distinct solutions to (2.5). By the first part of the proof these solutions are homogeneous, which implies that there are at least $\ell(t)$ distinct solutions at the point $M_{t'}^*(w)$. Thus $\ell(t) \leq \ell(t')$. By symmetry we also have $\ell(t') \leq \ell(t)$, which completes the proof. \square

Corollary 2.4. *Let $P \in C^\infty(T^*(X) \setminus 0)$ be an $N \times N$ system, homogeneous of degree m , and let $K \subset T^*(X) \setminus 0$ be a compact set. Suppose that $w \mapsto \lambda(w)$ is a section of eigenvalues of P with constant algebraic multiplicity for w in*

$$K_\varepsilon = \{w \in T^*(X) \setminus 0 : \inf_{w_0 \in K} |w - w_0| < \varepsilon\},$$

with distance given in terms of some fixed Riemannian metric. Then $\lambda(w)$ has constant algebraic multiplicity in the cone

$$\Gamma = \{M_t(w) : t > 0, w \in K_\varepsilon\}.$$

Proof. Let $M_t(w) \in \Gamma$ and suppose that the algebraic multiplicity of $\lambda(w)$ equals k for $w \in K_\varepsilon$. By assumption we then have $|P(w) - \lambda \text{Id}_N| = (\lambda - \lambda(w))^k e(w, \lambda)$, where $e(w, \lambda(w)) \neq 0$. Consider now equation (2.5). By the homogeneity of P this equation is equivalent to $|P(w) - t^{-m} \lambda \text{Id}_N| = 0$ for any $t > 0$. Since the left-hand side equals $(t^{-m} \lambda - \lambda(w))^k e(w, t^{-m} \lambda)$ and $\lambda(w) = t^{-m} M_t^* \lambda(w)$ by Proposition 2.3, this shows that $\lambda = M_t^* \lambda(w)$ is a solution to (2.5) of at least multiplicity k .

Using homogeneity again we find that for $\lambda = M_t^* \lambda(w)$ we have $e(w, t^{-m} \lambda) = e(w, \lambda(w)) \neq 0$, which shows that the multiplicity is precisely k . Since $M_t(w) \in \Gamma$ was arbitrary, the proof is complete. \square

In view of Corollary 2.4 we shall sometimes permit us to say that a system P has constant characteristics near a conic set $K \subset T^*(X) \setminus 0$ if it is clear from the context what we mean. Suppose now that $P(w)$ is homogeneous of degree m and of principal type with constant characteristics near a compact set $K \subset T^*(X) \setminus 0$ contained in the characteristic set $\Sigma(P) = \{w : |P(w)| = 0\}$ of $P(w)$. Let $\lambda(w)$ be the unique section of eigenvalues of $P(w)$ near K satisfying $\lambda(w) = 0$ for $w \in K$. By Definition 2.2 together with [4, Proposition 2.10] we then have $d\lambda(w) \neq 0$ in K , which in view of Proposition 2.3 implies that $d\lambda(w) \neq 0$ in a conic neighborhood of K . In particular, this means that for systems of principal type with constant characteristics, the section of eigenvalues close to the origin is a uniquely defined C^∞ function with non-vanishing differential, so the semi-bicharacteristics of the eigenvalues are well defined near the characteristic set $\Sigma(P)$. This makes the following definition possible.

Definition 2.5. We say that the $N \times N$ system $P \in \Psi_{\text{cl}}^m(X)$ of principal type and constant characteristics satisfies condition (Ψ) if the eigenvalues of the principal symbol satisfies condition (Ψ) .

Similarly, by the previous discussion it follows that the condition that the Hamilton vector field of an eigenvalue λ does not have the radial direction when $\lambda = 0$ is also well defined. Under this additional assumption, the section of eigenvalues close to the origin is then a uniquely defined homogeneous C^∞ function of principal type. In fact, if Definition 2.1 is changed to include the additional condition discussed in the remark following the definition, then the characterization of systems of principal type given by [4, Proposition 2.10] takes the following form. This is included only for the sake of completeness and will not be used here.

Proposition 2.6. *Let $P(w) \in C^\infty(T^*(X) \setminus 0)$ be an $N \times N$ system such that $|P(w_0)| = 0$, and let $\Upsilon(P)$ be given by (2.4). Assume that*

$$\{w_0\} \times \{0\} \cap \Upsilon(P) = \emptyset.$$

Let $\lambda(w) \in C^\infty$ be the unique section of eigenvalues of $P(w)$ satisfying $\lambda(w_0) = 0$. If ω is the canonical one form then $P(w)$ satisfies Definition 2.1 for some tangent vector $\partial_\nu \in T_{w_0}(T^(X) \setminus 0)$ such that $\langle \partial_\nu, \omega(w_0) \rangle = 0$ if and only if the Hamilton vector field $H_\lambda(w_0)$ is not proportional to the radial vector field at w_0 and the geometric multiplicity of the eigenvalue λ is equal to the algebraic multiplicity at w_0 .*

Note that as suggested in the statement of the proposition, the hypotheses $|P(w_0)| = 0$ and $\{w_0\} \times \{0\} \cap \Upsilon(P) = \emptyset$ imply that the section of eigenvalues $\lambda(w)$ of $P(w)$ satisfying $\lambda(w_0) = 0$ is a uniquely defined C^∞ function in a neighborhood of w_0 according to the discussion following Definition 2.2.

Proof. Inspecting the beginning of the proof of [4, Proposition 2.10] we conclude that the same arguments show that $P(w)$ satisfies Definition 2.1 for some tangent vector $\partial_\nu \in T_{w_0}(T^*(X) \setminus 0)$ such that $\langle \partial_\nu, \omega(w_0) \rangle = 0$ if and only if

$$\partial_\nu^k |P(w_0)| \neq 0, \quad \langle \partial_\nu, \omega(w_0) \rangle = 0, \quad k = \dim \text{Ker } P(w_0).$$

Now, if $P(w)$ is of principal type at w_0 then the geometric multiplicity k of λ is equal to the algebraic multiplicity m at w_0 by [4, Proposition 2.10]. Thus

$$\partial_\nu^m |P(w_0)| \neq 0, \quad |P(w) - \lambda \text{Id}_N| = (\lambda(w) - \lambda)^m e(w, \lambda)$$

for w in a neighborhood of w_0 where $e(w, \lambda) \neq 0$. Setting $\lambda = 0$ we obtain $0 \neq \partial_\nu^m |P(w_0)| = (\partial_\nu \lambda(w_0))^m e(w_0, 0)$. If $\langle \partial_\nu, \omega(w_0) \rangle = 0$ and $d\lambda(w_0) = \mu \omega(w_0)$ at w_0 for some $\mu \in \mathbb{C}$, then $0 \neq \partial_\nu \lambda(w_0) = \mu \langle \partial_\nu, \omega(w_0) \rangle = 0$, a contradiction.

To prove sufficiency, we note that if $H_\lambda(w_0)$ is not proportional to the radial vector field at w_0 then we can find a tangent vector $\partial_\nu \in T_{w_0}(T^*(X) \setminus 0)$ such that $\langle \partial_\nu, d\lambda(w_0) \rangle \neq 0$ and $\langle \partial_\nu, \omega(w_0) \rangle = 0$. But this gives $\partial_\nu^m |P(w_0)| \neq 0$ where m equals the algebraic and geometric multiplicity at w_0 , so by the first paragraph we conclude that $P(w)$ satisfies Definition 2.1 for a tangent vector ∂_ν such that $\langle \partial_\nu, \omega(w_0) \rangle = 0$. This completes the proof. \square

3. MINIMAL BICHARACTERISTICS

The purpose of this section is to recall the geometry that occurs when condition (Ψ) is violated. For a more thorough discussion as well as proofs for the results below we refer the reader to [16, Section 2], on which the following review is based.

Let us first fix some terminology. If $\gamma \subset T^*(X)$ is a curve with a parametrization $t \mapsto \gamma(t)$ defined (at least) for $a \leq t \leq b$, we shall say that $\text{Im } qp$ changes sign from $-$ to $+$ on γ if

$$(3.1) \quad \text{Im } qp(\gamma(a)) < 0 < \text{Im } qp(\gamma(b)).$$

If $\gamma|_{[a', b']}$ is the restriction of γ to $[a', b']$ and we have

- i) $\text{Im } qp(\gamma(t)) = 0$ for $a' \leq t \leq b'$,
- ii) for every $\varepsilon > 0$ one can find $a' - \varepsilon < s_- < a'$ and $b' < s_+ < b' + \varepsilon$ such that $\text{Im } qp(\gamma(s_-)) < 0 < \text{Im } qp(\gamma(s_+))$,

then we shall say that $\text{Im } qp$ *strongly* changes sign from $-$ to $+$ on $\gamma|_{[a', b']}$. If p and q are smooth homogeneous functions and γ is a bicharacteristic of $\text{Re } qp$ where $q \neq 0$ and (3.1) holds, then we can always find a subinterval of γ where $\text{Im } qp$ strongly changes sign from $-$ to $+$ by [16, Lemma 2.5].

Consider now the case where $p \in C^\infty(T^*(\mathbb{R}^n) \setminus 0)$ satisfies $\text{Re } p = \xi_1$. If $\gamma = I \times \{w_0\}$, $I = [a, b]$, we shall by $|\gamma|$ denote the usual arc length in \mathbb{R}^{2n} , so that $|\gamma| = b - a$. Furthermore, we will assume that all curves are bicharacteristics of $\text{Re } p = \xi_1$, that is, $w_0 = (x', 0, \xi') \in \mathbb{R}^{2n-1}$. We shall then employ the following notation.

Definition 3.1. Let $\gamma = [a, b] \times \{w_0\}$, and let $\gamma_j = [a_j, b_j] \times \{w_j\}$. If $\lim_{j \rightarrow \infty} w_j = w_0$, $\liminf_{j \rightarrow \infty} a_j \geq a$ and $\limsup_{j \rightarrow \infty} b_j \leq b$, then we shall write $\gamma_j \dashrightarrow \gamma$ as $j \rightarrow \infty$. If in addition $\lim_{j \rightarrow \infty} a_j = a$ and $\lim_{j \rightarrow \infty} b_j = b$ then we shall write $\gamma_j \rightarrow \gamma$ as $j \rightarrow \infty$.

Definition 3.2. If γ is a bicharacteristic of $\text{Re } p = \xi_1$ and there exists a sequence $\{\gamma_j\}_{j=1}^\infty$ of bicharacteristics of $\text{Re } p$ such that $\text{Im } p$ strongly changes sign from $-$ to $+$ on γ_j for all j and $\gamma_j \dashrightarrow \gamma$ as $j \rightarrow \infty$, we set

$$L_p(\gamma) = \inf \{ \liminf_{j \rightarrow \infty} |\gamma_j| : \gamma_j \dashrightarrow \gamma \text{ as } j \rightarrow \infty \},$$

where the infimum is taken over all such sequences. We shall write $L_p(\gamma) \geq 0$ to signify the existence of such a sequence $\{\gamma_j\}_{j=1}^\infty$.

Note that the definition of $L_p(\gamma)$ corresponds to what is denoted by L_0 in [11, p. 97], when $\gamma = [a, b] \times \{w_0\}$ is given by

$$a \leq x_1 \leq b, \quad x' = (x_2, \dots, x_n) = 0, \quad \xi = \varepsilon_n,$$

and $\operatorname{Im} p(a, w_0) < 0 < \operatorname{Im} p(b, w_0)$. For a proof of this claim, see the remark following [16, Definition 2.9]. Here $\varepsilon_n = (0, \dots, 0, 1) \in \mathbb{R}^n$, and we shall in what follows write ξ^0 in place of ε'_n . Note also that if $L_p(\gamma)$ exists, then $L_p(\gamma) \leq |\gamma|$ by definition. Moreover, if $\operatorname{Im} p$ strongly changes sign from $-$ to $+$ on γ then it is easy to see that the conditions of Definition 3.2 are satisfied.

We now recall the definition of a minimal bicharacteristic.

Definition 3.3. Let $I \subset \mathbb{R}$ be a compact interval possibly reduced to a point and let $\tilde{\gamma} : I \rightarrow T^*(X) \setminus 0$ be a characteristic point or a compact one dimensional bicharacteristic interval of the homogeneous function $p \in C^\infty(T^*(X) \setminus 0)$. Suppose that there exists a function $q \in C^\infty(T^*(X) \setminus 0)$ and a C^∞ homogeneous canonical transformation χ from an open conic neighborhood V of

$$\Gamma = \{(x_1, 0, \varepsilon_n) : x_1 \in I\} \subset T^*(\mathbb{R}^n)$$

to an open conic neighborhood $\chi(V) \subset T^*(X) \setminus 0$ of $\tilde{\gamma}(I)$ such that

- (i) $\chi(x_1, 0, \varepsilon_n) = \tilde{\gamma}(x_1)$ and $\operatorname{Re} \chi^*(qp) = \xi_1$ in V ,
- (ii) $L_{\chi^*(qp)}(\Gamma) = |\Gamma|$.

Then we say that $\tilde{\gamma}(I)$ is a minimal characteristic point or a minimal bicharacteristic interval if $|I| = 0$ or $|I| > 0$, respectively.

The definition of the arclength is of course dependent of the choice of Riemannian metric on $T^*(\mathbb{R}^n)$. However, since we are only using the arclength to compare curves where one is contained within the other and both are parametrizable through condition (i), the results here and Definition 3.3 in particular are independent of the chosen metric.

Some comments on the implications of Definition 3.3 are in order. First, note that condition (i) implies that $q \neq 0$ and $\operatorname{Re} H_{qp} \neq 0$ on $\tilde{\gamma}$, and that by definition, a minimal bicharacteristic interval is a compact one dimensional bicharacteristic interval (see [11, Definition 26.4.9]). If $\operatorname{Im} qp$ changes sign from $-$ to $+$ on a bicharacteristic $\gamma \subset T^*(X) \setminus 0$ of $\operatorname{Re} qp$ where $q \neq 0$, then we can always find a minimal characteristic point $\tilde{\gamma} \in \gamma$ or a minimal bicharacteristic interval $\tilde{\gamma} \subset \gamma$. In the language of [11, Section 26.4], $\tilde{\gamma}$ is the subset of γ with the property that $\operatorname{Im} qp$ changes sign from $-$ to $+$ on bicharacteristics of $\operatorname{Re} qp$ arbitrarily close to $\tilde{\gamma}$. For a proof of this fact, see [11, p. 97] or the discussion preceding [16, Proposition 2.12]. In fact, we have the following result.

Proposition 3.4. *Let $\gamma = [a, b] \times \{w_0\}$ be a bicharacteristic of $\operatorname{Re} p = \xi_1$, and assume that $L(\gamma) \geq 0$. Then there exists a minimal characteristic point $\Gamma \in \gamma$ of p or a minimal bicharacteristic interval $\Gamma \subset \gamma$ of p of length $L(\gamma)$ if $L(\gamma) = 0$ or $L(\gamma) > 0$, respectively. If $\Gamma = [a_0, b_0] \times \{w_0\}$ and $a_0 < b_0$, that is, $L(\gamma) > 0$, then*

$$\operatorname{Im} p_{(\alpha)}^{(\beta)}(t, w_0) = 0$$

for all α, β with $\beta_1 = 0$ if $a_0 \leq t \leq b_0$. Conversely, if γ is a minimal characteristic point or a minimal bicharacteristic interval then $L(\gamma) = |\gamma|$.

Proof. See the proof of [16, Proposition 2.12]. □

Keeping the notation from Definition 3.3, we note in view of Proposition 3.4 that condition (ii) implies that there exists a sequence $\{\Gamma_j\}_{j=1}^\infty$ of bicharacteristics of $\operatorname{Re} \chi^*(qp)$ on which $\operatorname{Im} \chi^*(qp)$ strongly changes sign from $-$ to $+$, such that $\Gamma_j \rightarrow \Gamma$ as $j \rightarrow \infty$. By our choice of terminology, the sequence $\{\Gamma_j\}_{j=1}^\infty$ may simply be a sequence of points when $L(\Gamma) = 0$. Conversely, if $\{\Gamma_j\}_{j=1}^\infty$ is a point sequence then $L(\Gamma) = 0$. Also note that if $\tilde{\gamma}(I)$ is minimal, and condition (i) in Definition 3.3 is satisfied for some other choice of maps q', χ' , then condition (ii) also holds for q', χ' ; in other words,

$$L_{\chi^*(qp)}(\Gamma) = |\Gamma| = L_{(\chi')^*(q'p)}(\Gamma).$$

This follows by an application of Proposition 3.4 together with [11, Lemma 26.4.10]. It is then also clear that $\tilde{\gamma}(I)$ is a minimal characteristic point or a minimal bicharacteristic interval of the homogeneous function $p \in C^\infty(T^*(X) \setminus 0)$ if and only if $\Gamma(I)$ is a minimal characteristic point or a minimal bicharacteristic interval of $\chi^*(qp) \in C^\infty(T^*(\mathbb{R}^n) \setminus 0)$ for any maps q and χ satisfying condition (i) in Definition 3.3.

Definition 3.5. A minimal bicharacteristic interval $\Gamma = [a_0, b_0] \times \{w_0\} \subset T^*(\mathbb{R}^n) \setminus 0$ of the homogeneous function $p = \xi_1 + i \operatorname{Im} p$ of degree 1 is said to be ϱ -minimal if there exists a $\varrho \geq 0$ such that $\operatorname{Im} p$ vanishes in a neighborhood of $[a_0 + \kappa, b_0 - \kappa] \times \{w_0\}$ for any $\kappa > \varrho$.

By a 0-minimal bicharacteristic interval Γ we thus mean a minimal bicharacteristic interval such that the imaginary part vanishes in a neighborhood of any proper closed subset of Γ . Note that this does not hold for minimal bicharacteristic intervals in general. However, the following result does hold, which concludes this section.

Theorem 3.6. *If Γ is a minimal bicharacteristic interval in $T^*(\mathbb{R}^n) \setminus 0$ of the homogeneous function $p = \xi_1 + i \operatorname{Im} p$ of degree 1, where the imaginary part is independent of ξ_1 , then there exists a sequence $\{\Gamma_j\}_{j=1}^\infty$ of ϱ_j -minimal bicharacteristic intervals of p such that $\Gamma_j \rightarrow \Gamma$ and $\varrho_j \rightarrow 0$ as $j \rightarrow \infty$.*

Proof. See the proof of [16, Theorem 2.18]. □

4. SOLVABILITY AND MICROLOCAL INCLUSION RELATIONS

If $u = (u_j)$ and $v = (v_j)$ are vectors in \mathbb{C}^N with u_j and v_j in $L^2(X, \mathbb{C})$ for $1 \leq j \leq N$, let

$$(4.1) \quad (u, v)_{L^2(X, \mathbb{C}^N)} = \sum_{j=1}^N (u_j, v_j)$$

where (\cdot, \cdot) denotes the usual scalar product on $L^2(X, \mathbb{C})$. Recall that the Sobolev space $H_{(s)}(X, \mathbb{C})$, $s \in \mathbb{R}$, is a local space, that is, if $\varphi \in C_0^\infty(X, \mathbb{C})$ and $\psi \in H_{(s)}(X, \mathbb{C})$ then $\varphi\psi \in H_{(s)}(X, \mathbb{C})$, and the corresponding operator of multiplication is continuous. If $\|\cdot\|_{(s)}$ is the usual norm on $H_{(s)}(X, \mathbb{C})$ we shall with abuse of notation let $H_{(s)}(X, \mathbb{C}^N)$ be the space of distributions $u = (u_j) \in \mathcal{D}'(X, \mathbb{C}^N)$ such that $u_j \in H_{(s)}(X, \mathbb{C})$ for $1 \leq j \leq N$, equipped with the norm

$$\|u\|_{(s)} = \left(\sum_{j=1}^N \|u_j\|_{(s)}^2 \right)^{1/2}.$$

Thus we can define

$$H_{(s)}^{\text{loc}}(X, \mathbb{C}^N) = \{u \in \mathcal{D}'(X, \mathbb{C}^N) : \varphi u \in H_{(s)}(X, \mathbb{C}^N), \forall \varphi \in C_0^\infty(X, \mathbb{C})\}.$$

This is a Fréchet space, and its dual with respect to the pairing (4.1) is

$$H_{(-s)}^{\text{comp}}(X, \mathbb{C}^N) = H_{(-s)}^{\text{loc}}(X, \mathbb{C}^N) \cap \mathcal{E}'(X, \mathbb{C}^N).$$

Recall also that the wave front set of $u = (u_j) \in \mathcal{D}'(X, \mathbb{C}^N)$ is defined as the union of $WF(u_j)$. For a system A of pseudodifferential operators in X we shall as usual let $WF(A)$ be the smallest closed conic set in $T^*(X) \setminus 0$ such that $A \in \Psi^{-\infty}$ in the complement.

Definition 4.1. If $K \subset T^*(X) \setminus 0$ is a compactly based cone we shall say that the range of the $N \times N$ system $Q \in \Psi_{\text{cl}}^k(X)$ is microlocally contained in the range of the $N \times N$ system $P \in \Psi_{\text{cl}}^m(X)$ at K if there exists an integer N_0 such that for every $f \in H_{(N_0)}^{\text{loc}}(X, \mathbb{C}^N)$ one can find a $u \in \mathcal{D}'(X, \mathbb{C}^N)$ with $WF(Pu - Qf) \cap K = \emptyset$.

If $\text{Id}_N \in \Psi_{\text{cl}}^0(X)$ is the identity $\text{Id}_N : u \mapsto u \in \mathcal{D}'(X, \mathbb{C}^N)$ then we obtain from Definition 4.1 the definition of microlocal solvability for a system of pseudodifferential operators (see [11, Definition 26.4.3] and the discussion following equation (1.1) in [2]) by setting $Q = \text{Id}_N$. Thus, the range of the identity is microlocally contained in the range of P at K if and only if P is microlocally solvable at K . Note also that if P and Q satisfy Definition 4.1 for some integer N_0 , then due to the inclusion

$$H_{(t)}^{\text{loc}}(X, \mathbb{C}^N) \subset H_{(s)}^{\text{loc}}(X, \mathbb{C}^N), \quad \text{if } s < t,$$

the statement also holds for any integer $N' \geq N_0$. Hence N_0 can always be assumed to be positive. Furthermore, the property is preserved if Q is composed with a properly supported $N \times N$ system $Q_1 \in \Psi_{\text{cl}}^{k'}(X)$ from the right. Indeed, let g be an arbitrary element in $H_{(N_0+k')}^{\text{loc}}(X, \mathbb{C}^N)$. Then $f = Q_1 g \in H_{(N_0)}^{\text{loc}}(X, \mathbb{C}^N)$ since Q_1 is continuous

$$Q_1 : H_{(s)}^{\text{loc}}(X, \mathbb{C}^N) \rightarrow H_{(s-k')}^{\text{loc}}(X, \mathbb{C}^N)$$

for every $s \in \mathbb{R}$, so by Definition 4.1 there exists a $u \in \mathcal{D}'(X, \mathbb{C}^N)$ with $WF(Pu - QQ_1 f) \cap K = \emptyset$. Hence the range of QQ_1 is microlocally contained in the range of P at K with the integer N_0 replaced by $N_0 + k'$.

The property given by Definition 4.1 is also preserved under composition of both P and Q with a properly supported $N \times N$ system from the left. In view of (1.1) this follows immediately from the fact that properly supported scalar pseudodifferential operators are microlocal, that is,

$$WF(Au) \subset WF(u) \cap WF(A), \quad u \in \mathcal{D}'(X).$$

Just as microlocal solvability of a pseudodifferential operator P implies an a priori estimate for the adjoint P^* , we have the following result for systems satisfying Definition 4.1.

Lemma 4.2. *Let $K \subset T^*(X) \setminus 0$ be a compactly based cone. Let $Q \in \Psi_{\text{cl}}^k(X)$ and $P \in \Psi_{\text{cl}}^m(X)$ be properly supported $N \times N$ systems such that the range of Q is microlocally contained in the range of P at K . If $Y \Subset X$ satisfies $K \subset T^*(Y)$ and if N_0 is the integer in Definition 4.1, then for every positive integer κ we can find a constant C , a positive integer ν and a properly supported $N \times N$ system A with $WF(A) \cap K = \emptyset$ such that*

$$(4.2) \quad \|Q^*v\|_{(-N_0)} \leq C(\|P^*v\|_{(\nu)} + \|v\|_{(-N_0-\kappa-n)} + \|Av\|_{(0)})$$

for all $v \in C_0^\infty(Y)$.

By replacing the range \mathbb{C} by \mathbb{C}^N , the proof of the corresponding result for the scalar case (see [16, Lemma 2.3]) can be used without additional changes to prove Lemma 4.2. We omit the details. Note also that since (4.2) holds for any κ , it is actually superfluous to include the dimension n in the norm $\|v\|_{(-N_0-\kappa-n)}$. However, for our purposes, it turns out that this is the most convenient formulation.

We will need the following analogue of [11, Proposition 26.4.4]. Since the proof again is the same as for the corresponding result for scalar operators, we refer to the notation and proof of [16, Proposition 2.4] for details.

Proposition 4.3. *Let $K \subset T^*(X) \setminus 0$ and $K' \subset T^*(Y) \setminus 0$ be compactly based cones and let χ be a homogeneous symplectomorphism from a conic neighborhood of K' to one of K such that $\chi(K') = K$. Let $A \in I^{m'}(X \times Y, \Gamma')$ and $B \in I^{m''}(Y \times X, (\Gamma^{-1})')$ where Γ is the graph of χ , and assume that the $N \times N$ systems A and B are properly supported and non-characteristic at the restriction of the graphs of χ and χ^{-1} to K' and to K respectively, while $WF'(A)$ and $WF'(B)$ are contained in small conic neighborhoods. Then the range of the $N \times N$ system Q of pseudodifferential operators in X is microlocally contained in the range of the $N \times N$ system P of pseudodifferential operators in X at K if and only if the range of the system BQA in Y is microlocally contained in the range of the system BPA in Y at K' .*

It will be convenient to record the following result, concerning necessary conditions for semi-global solvability for systems of principal type and constant characteristics, using the notion of minimal bicharacteristics. Note that this theorem therefore in a sense corresponds to [11, Theorem 26.4.7'] in the scalar case.

Theorem 4.4. *Let $P \in \Psi_{cl}^m(X)$ be a properly supported $N \times N$ system of pseudodifferential operators of principal type in the open conic set $\Omega \subset T^*(X) \setminus 0$. Let P_m be the homogeneous principal symbol of P , and let $I = [a_0, b_0] \subset \mathbb{R}$ be a compact interval possibly reduced to a point. Let $\gamma : I \rightarrow \Omega$ be a curve belonging to the characteristic set $\Sigma(P_m)$ of P_m , and suppose that P has constant characteristics near $\gamma(I)$. If $\lambda(w)$ is the unique section of eigenvalues of $P_m(w)$ satisfying $\lambda \circ \gamma = 0$, and γ is either*

- (a) *a minimal characteristic point of $\lambda(w)$, or*
- (b) *a minimal bicharacteristic interval of $\lambda(w)$ with injective regular projection in $S^*(X)$,*

then P is not solvable at the cone generated by $\gamma(I)$.

We wish to point out that although case (b) is not explicitly treated in [4], this result is essentially contained in [4, Theorem 2.7]. In fact, for systems of principal type and constant characteristics, [4, Theorem 2.7] says that condition (1.3) is equivalent to microlocal solvability near a point $w_0 \in T^*(X) \setminus 0$ under the additional assumption that the Hamilton vector field H_λ of the section of eigenvalues of $P_m(w)$ near w_0 satisfying $\lambda(w_0) = 0$ does not have the radial direction at w_0 . If $\gamma(I)$ satisfies property (a), then Definition 3.3 implies that H_λ is not proportional to the radial vector field at $\gamma(I)$ and that (1.3) cannot hold in any neighborhood of $\gamma(I)$, and since the wave front set is conic by definition it follows by [4, Theorem 2.7] that P is not solvable at the cone generated by $\gamma(I)$. Hence, it only remains to verify Theorem 4.4 in the case when γ satisfies property (b). However, note that after locally preparing the system P to a suitable normal form by means

of [4, Lemma 4.1], the necessity part of [4, Theorem 2.7] is proved by repetition of the Hörmander–Moyer proof of the necessity of condition (Ψ) for semi-global solvability for scalar operators (see the proof of [11, Theorem 26.4.7]). By for example extending the preparation result [4, Lemma 4.1] to a neighborhood of a one dimensional bicharacteristic interval as discussed in Section 5 below, the same arguments therefore show that condition (Ψ) is necessary also for semi-global solvability for systems of principal type and constant characteristics. For completeness, we have included a short proof of Theorem 4.4, which can be found in Section 6.

We also mention that if $P \in \Psi_{\text{cl}}^m(X)$ is an $N \times N$ system of principal type and constant characteristics that is not microlocally solvable in any neighborhood of a point $w_0 \in T^*(X) \setminus 0$, and the Hamilton vector field H_λ of the section of eigenvalues of $P_m(w)$ near w_0 satisfying $\lambda(w_0) = 0$ does not have the radial direction at w_0 , then $\lambda(w)$ fails to satisfy condition (Ψ) in every neighborhood of w_0 by [4, Theorem 2.7]. In view of the alternative version of condition (1.3) given by [11, Theorem 26.4.12], it is then easy to see using [10, Theorem 21.3.6] and [11, Lemma 26.4.10] that w_0 is a minimal characteristic point of $\lambda(w)$.

If γ is a minimal bicharacteristic interval of a function $\lambda(w)$ of principal type such that γ is contained in a curve along which $\lambda(w)$ fails to satisfy condition (1.3), then γ has injective regular projection in $S^*(X)$ by the proof of [11, Theorem 26.4.12]. Since solvability at a conic set $K \subset T^*(X) \setminus 0$ implies solvability at any smaller closed cone, the discussion preceding Proposition 3.4 therefore yields the following corollary to Theorem 4.4, corresponding to [11, Theorem 26.4.7].

Corollary 4.5. *Let $P \in \Psi_{\text{cl}}^m(X)$ be a properly supported $N \times N$ system of pseudodifferential operators of principal type in the open conic set $\Omega \subset T^*(X) \setminus 0$. Let P_m be the homogeneous principal symbol of P , and let $I = [a_0, b_0] \subset \mathbb{R}$ be a compact interval not reduced to a point. Let $\gamma : I \rightarrow \Omega$ be a curve containing a point $\gamma(t_0) \in \Sigma(P_m)$, and suppose that P has constant characteristics near $\gamma(I)$. If $\varepsilon > 0$ is the number given by Definition 2.2 and $\lambda(w)$ is the unique section of eigenvalues of P_m satisfying $\lambda(\gamma(t_0)) = 0$, assume that $|\lambda \circ \gamma(t)| < \varepsilon$ for $a_0 \leq t \leq b_0$ so that $\lambda(w)$ is a uniquely defined C^∞ function in a neighborhood of $\gamma(I)$. If there is a homogeneous C^∞ function q in $T^*(X) \setminus 0$ such that γ is a bicharacteristic interval of $\text{Re } q\lambda$ where $\text{Re } H_{q\lambda} \neq 0$ and*

$$\text{Im } q\lambda(\gamma(a_0)) < 0 < \text{Im } q\lambda(\gamma(b_0)),$$

then P is not solvable at the cone generated by $\gamma(I)$.

We now proceed to the main result of the paper, generalizing [16, Theorem 2.19] to systems of principal type and constant characteristics for which Theorem 4.4 implies non-solvability.

Theorem 4.6. *Let $K \subset T^*(X) \setminus 0$ be a compactly based cone. Let $P \in \Psi_{\text{cl}}^m(X)$ and $Q \in \Psi_{\text{cl}}^k(X)$ be properly supported $N \times N$ systems of pseudodifferential operators such that the range of Q is microlocally contained in the range of P at K , where P is system of principal type and constant characteristics near K . Let P_m be the homogeneous principal symbol of P , and let $I = [a_0, b_0] \subset \mathbb{R}$ be a compact interval possibly reduced to a point. Suppose that $\gamma : I \rightarrow T^*(X) \setminus 0$ belongs to the characteristic set $\Sigma(P_m)$ of P_m and that K contains a conic neighborhood of $\gamma(I)$. If $\lambda(w)$ is the unique section of eigenvalues of $P_m(w)$ satisfying $\lambda \circ \gamma = 0$, and γ is either*

- (a) a minimal characteristic point of $\lambda(w)$, or
- (b) a minimal bicharacteristic interval of $\lambda(w)$ with injective regular projection in $S^*(X)$,

then there exists an $N \times N$ system $E \in \Psi_{\text{cl}}^{k-m}(X)$ such that the terms in the asymptotic expansion of the symbol of $Q - PE$ vanish of infinite order at $\gamma(I)$.

Note that when proving Theorem 4.6 we may assume that P and Q have the same order. In fact, let $Q_1 \in \Psi_{\text{cl}}^{m-k}(X)$ be a properly supported, elliptic $N \times N$ system. By the discussion following Definition 4.1 we have that the range of QQ_1 is microlocally contained in the range of P at K . None of the other assumptions in Theorem 4.6 are affected by this composition, so suppose that the theorem is proved for operators of the same order. Since both P and QQ_1 have order m , we can then find a system $E \in \Psi_{\text{cl}}^0(X)$ such that all the terms in the asymptotic expansion of the symbol of $QQ_1 - PE$ vanish of infinite order at $\gamma(I)$. If $Q_1^{-1} \in \Psi_{\text{cl}}^{k-m}(X)$ is a properly supported parametrix of Q_1 , the calculus then gives that all the terms in the asymptotic expansion of the symbol of

$$(QQ_1 - PE) \circ Q_1^{-1} \equiv Q - PEQ_1^{-1} \pmod{\Psi^{-\infty}}$$

vanish of infinite order at $\gamma(I)$. Thus Theorem 4.6 holds with E replaced by $EQ_1^{-1} \in \Psi_{\text{cl}}^{k-m}(X)$.

We postpone the proof of Theorem 4.6 and instead show that Theorem 4.6 has applications to scalar non-principal type pseudodifferential operators. For a similar example related to solvability, see [4, Theorem 2.9]. If L is a properly supported scalar operator we shall in this context let L^m denote the composition $L \circ \dots \circ L$ with L occurring m times, while L^0 is understood to be the identity $\text{Id} \in \Psi_{\text{cl}}^0(X)$.

Theorem 4.7. *Let $K \subset T^*(X) \setminus 0$ be a compactly based cone. Let $L \in \Psi_{\text{cl}}^1(X)$ be a properly supported scalar operator of principal type near K , and let $A_j \in \Psi_{\text{cl}}^0(X)$ for $0 \leq j < N$ be properly supported scalar operators. If $P \in \Psi_{\text{cl}}^N(X)$ is the operator*

$$Pu = L^N u + \sum_{j=0}^{N-1} A_j L^j u,$$

let $Q \in \Psi_{\text{cl}}^k(X)$ be properly supported and assume that the range of Q is microlocally contained in the range of P at K . Let $w \mapsto \lambda(w)$ be the homogeneous principal symbol of L , and let $I = [a_0, b_0] \subset \mathbb{R}$ be a compact interval possibly reduced to a point. Suppose that K contains a conic neighborhood of $\gamma(I)$, where $\gamma : I \rightarrow T^*(X) \setminus 0$ is either

- (a) a minimal characteristic point of $\lambda(w)$, or
- (b) a minimal bicharacteristic interval of $\lambda(w)$ with injective regular projection in $S^*(X)$.

Then there exists a properly supported scalar operator $E \in \Psi_{\text{cl}}^{k-1}(X)$ such that the terms in the asymptotic sum of the symbol of $Q - PE$ vanish of infinite order at $\gamma(I)$.

Proof. This is a standard reduction to a first order system. If \mathcal{Q} is the $N \times N$ system given by the block form

$$\mathcal{Q} = \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix} \in \Psi_{\text{cl}}^k(X),$$

then the range of \mathcal{Q} is microlocally contained in the range of \mathcal{P} at K , where

$$\mathcal{P} = \begin{pmatrix} L & -1 & 0 & \dots & 0 \\ 0 & L & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_0 & A_1 & A_2 & \dots & A_{N-1} + L \end{pmatrix} \in \Psi_{\text{cl}}^1(X).$$

Indeed, if N_0 is the integer given by Definition 4.1, let $f \in H_{(N_0)}^{\text{loc}}(X, \mathbb{C}_N)$ be given by $f = {}^t(f_1, \dots, f_N)$. Then we can find a scalar distribution $u \in \mathcal{D}'(X)$ such that $WF(Pu - Qf_N) \cap K = \emptyset$. Now let $v_{j+1} = L^j u$ for $0 \leq j < N$ and set $v = {}^t(v_1, \dots, v_N)$. Then $\mathcal{Q}f = {}^t(0, \dots, 0, Qf_N)$ and $\mathcal{P}v = {}^t(0, \dots, 0, Pu)$, which proves the claim. Since $\lambda(w)$ is the only section of eigenvalues of the principal symbol of \mathcal{P} , we find by an application of [4, Proposition 2.10] that \mathcal{P} is a system of principal type and constant characteristics near K . By Theorem 4.6 there is an $N \times N$ system $\mathcal{B} = (B_{jk}) \in \Psi_{\text{cl}}^{k-1}(X)$ such that the terms in the asymptotic expansion of the symbol of $\mathcal{Q} - \mathcal{P} \circ \mathcal{B}$ vanish of infinite order at $\gamma(I)$. This means that the terms in the asymptotic expansions of the symbols of

- (i) $Q - A_0 B_{1N} - \dots - A_{N-1} B_{NN} - LB_{NN}$,
- (ii) $LB_{jN} - B_{(j+1)N}$, $1 \leq j < N$,

vanish of infinite order at $\gamma(I)$, which implies that the same holds for $Q - PB_{1N}$. Indeed, write

$$\begin{aligned} PB_{1N} &= L^{N-1}(LB_{1N} - B_{2N}) + \dots + L(LB_{(N-1)N} - B_{NN}) + LB_{NN} \\ &\quad + A_0 B_{1N} + \sum_{j=1}^{N-1} A_j \left(B_{(j+1)N} + \sum_{\ell=1}^j L^{j-\ell} (LB_{\ell N} - B_{(\ell+1)N}) \right) \\ &= LB_{NN} + \sum_{j=0}^{N-1} A_j B_{(j+1)N} + R, \end{aligned}$$

where R in view of (ii) and the calculus has a symbol with an asymptotic expansion whose terms vanish of infinite order at $\gamma(I)$. Since

$$Q - PB_{1N} = Q - LB_{NN} - \sum_{j=1}^N A_{j-1} B_{jN} - R,$$

the result now follows by (i) by setting $E = B_{1N}$. This completes the proof. \square

Keeping the notation from Theorem 4.7 and its proof, we remark that by comparing with the scalar principal type case we would expect the order of the operator $E \in \Psi_{\text{cl}}^{k-1}(X)$ to be lower. (E does have the expected order when $N = 1$, which is not surprising since P is of principal type then.) Since $Q \in \Psi_{\text{cl}}^k(X)$ it follows that if $N > 1$ then the terms $\sigma_j(PE)$ in the asymptotic expansion of the symbol of PE that are homogeneous of degree $k < j \leq N + k - 1$ must vanish of infinite order at $\gamma(I)$; these terms can be traced back to the operator R . Even though only the principal symbol is invariantly defined a priori, the statement has meaning in view of the symbol calculus, see [10, Theorem 18.1.17]. Since $d\lambda \neq 0$ near $\gamma(I)$, this means that the terms in the asymptotic expansion of the symbol of E that are homogeneous of degree $k - N < \ell \leq k - 1$ must vanish of infinite order at $\gamma(I)$. (Of course, since $\gamma(I)$ has empty interior, we cannot from this infer that $E \in \Psi^{k-N}$ at $\gamma(I)$ in the sense of the discussion preceding [10, Proposition 18.1.26].) Indeed,

if $w_0 \in \gamma(I)$ and $\sigma_E \sim e_{k-1} + e_{k-2} + \dots$ then the principal symbol of PE is $\sigma_{N+k-1}(PE) = \lambda^N e_{k-1}$, so e_{k-1} vanishes of infinite order at w_0 by Lemma A.4 in the appendix. If $k < j \leq N+k-1$ then the only term in $\sigma_j(PE)$ that does not involve the functions $e_{j-N+1}, \dots, e_{k-1}$ or their derivatives is $\lambda^N e_{j-N}$, so the claim follows by induction with respect to j and an application of Lemma A.4. This means that if q is the principal symbol of Q then

$$(4.3) \quad \partial_x^\alpha \partial_\xi^\beta (q(x, \xi) - \lambda^N(x, \xi) e_{k-N}(x, \xi))|_{(x, \xi) \in \gamma(I)} = 0 \quad \text{for all } \alpha, \beta \in \mathbb{N}^n,$$

since $\sigma_k(Q - PE)$ vanishes of infinite order, and the only term in $\sigma_k(PE)$ that does not involve the functions $e_{k-N+1}, \dots, e_{k-1}$ or their derivatives is $\lambda^N e_{k-N}$.

Note also that under the hypotheses of Theorem 4.7 it follows that P is not solvable at the cone generated by $\gamma(I)$. In the case when condition (a) holds, this is an immediate consequence of [4, Theorem 2.9] in view of the discussion following Theorem 4.4. If instead (b) holds, then \mathcal{P} fails to be solvable at the cone generated by $\gamma(I)$ by an application of Theorem 4.4. If P is solvable there, then the arguments in the proof of [4, Theorem 2.9] can be used to arrive at a contradiction. That is, given ${}^t(f_1, \dots, f_N)$ we set $u_1 = 0$, $u_2 = -f_1$ and recursively $u_{j+1} = Lu_j - f_j$ for $0 \leq j < N$. Then $\mathcal{P}({}^t(u_1, \dots, u_{N-1}, u_N)) = (f_1, \dots, f_{N-1}, f)$, with

$$f = Lu_N + \sum_{j=0}^{N-1} A_j u_{j+1} = - \sum_{\ell=1}^{N-1} L^{N-\ell} f_\ell - \sum_{j=0}^{N-1} \sum_{\ell=1}^j A_j L^{j-\ell} f_\ell.$$

If ${}^t(f_1, \dots, f_N)$ belongs to an appropriate local Sobolev space determined by the definition of solvability for P and the formula above, then there is a distribution $u \in \mathcal{D}'(X)$ such that $Pu - f - f_N$ has no wave front set in the cone generated by $\gamma(I)$. If we put $v_1 = u$ and recursively $v_{j+1} = Lv_j$ for $1 \leq j < N$ then $U = {}^t(u_1, \dots, u_N) + {}^t(v_1, \dots, v_N)$ satisfies $\mathcal{P}U = {}^t(f_1, \dots, f_N) + G$, where the wave front set of the vector G does not meet the cone generated by $\gamma(I)$, which is a contradiction.

If $P \in \Psi_{\text{cl}}^m(X)$ is a scalar operator we shall, for the rest of this section only, let $\text{Ran } P$ denote the range of P viewed as an operator $P : \mathcal{D}'(X) \rightarrow \mathcal{D}'(X)/C^\infty(X)$,

$$\text{Ran } P = \{f \in \mathcal{D}'(X) : f - Pu \in C^\infty(X) \text{ for some } u \in \mathcal{D}'(X)\}.$$

The operators L^j that appear in Theorem 4.7 enjoy the following property.

Corollary 4.8. *Let $L \in \Psi_{\text{cl}}^1(X)$ be a properly supported scalar operator, and assume that the hypotheses of Theorem 4.7 hold. Then we have the following chain of strict inclusions:*

$$\dots \subsetneq \text{Ran } L^{k+1} \subsetneq \text{Ran } L^k \subsetneq \dots \subsetneq \text{Ran } L \subsetneq \text{Ran Id}.$$

In particular, if j and k are non-negative integers, then $\text{Ran } L^j \subset \text{Ran } L^k$ if and only if $j \geq k$.

Proof. Let $k \geq 0$. If $f \in \text{Ran } L^{k+1}$, let $u \in \mathcal{D}'(X)$ satisfy $f - L^{k+1}u \in C^\infty$. Since L is continuous $L : \mathcal{D}'(X) \rightarrow \mathcal{D}'(X)$, we have $v = Lu \in \mathcal{D}'(X)$. Now $f - L^k v = f - L^{k+1}u \in C^\infty$, so $f \in \text{Ran } L^k$.

Conversely, assume to reach a contradiction that $\text{Ran } L^k \subset \text{Ran } L^{k+1}$, and let K be the cone given by Theorem 4.7 containing a minimal bicharacteristic $\gamma(I)$ of the principal symbol $\lambda(w)$ of L . It is clear that if $\text{Ran } L^k \subset \text{Ran } L^{k+1}$ then the range of L^k is microlocally contained in the range of L^{k+1} at K , so by an application of the theorem with $P = L^{k+1}$ and $Q = L^k$ we obtain an operator $E \in \Psi_{\text{cl}}^{k-1}(X)$

with symbol $e \sim e_{k-1} + e_{k-2} + \dots$ such that, in particular, the term $\sigma_k(Q - PE)$ in the asymptotic expansion of the symbol of $Q - PE$ that is homogeneous of degree k vanishes of infinite order at $w_0 \in \gamma(I)$. Since $\lambda(w)$ is assumed to be of principal type near K there is a tangent vector $\partial_\nu \in T_{w_0}(T^*(X) \setminus 0)$ such that $\partial_\nu \lambda(w_0) = \langle \partial_\nu, d\lambda \rangle \neq 0$. In view of (4.3) this implies that

$$0 = \partial_\nu^k (\lambda^k - \lambda^{k+1} e_{-1})|_{w_0} = k! (\partial_\nu \lambda(w_0))^k \neq 0,$$

a contradiction. If $k = 0$ this is to be interpreted as $0 = 1 - (\lambda(w_0))e_{-1}(w_0) = 1$, which also gives a contradiction. \square

Of course, we already know that $\text{Ran } L^j \subset \text{Ran } L$ implies that $j > 0$ under the hypotheses of Theorem 4.7. Indeed, in view of Definition 3.3 it follows by [11, Theorem 26.4.7'] together with [11, Proposition 26.4.4] that L fails to be solvable at the cone generated by $\gamma(I)$. In view of the discussion following Definition 4.1, the range of the identity is therefore not microlocally contained in the range of L at this cone, which shows that the inclusion $\text{Ran Id} \subset \text{Ran } L$ cannot hold.

5. PREPARATION

The purpose of this section is to prove a preparation result that will be used when proving Theorem 4.6. We first discuss when the kernel of a matrix valued function is a complex vector bundle.

Let X be a C^∞ manifold and $P(w)$ an $N \times N$ system varying smoothly with $w \in X$, and suppose that there is a unique section of eigenvalues $\lambda(w)$ of $P(w)$ vanishing along a compact and smooth simple curve $\gamma \subset \Sigma(P)$, where $\lambda(w)$ has constant multiplicity J in a neighborhood. Since the eigenvalues of $P(w)$ depend continuously on $w \in X$, it follows that there exists a neighborhood Y of γ and a small constant $c > 0$ such that the operator valued function

$$w \mapsto \Pi(w) = \frac{1}{2\pi i} \int_{|z|=c} (z \text{Id}_N - P(w))^{-1} dz \in C^\infty(Y)$$

is the projection onto the generalized eigenvectors for the eigenvalue $\lambda(w)$ of $P(w)$ (see for example [12, pp. 40–45]). The dimension of the algebraic eigenspace $\text{Ran } \Pi(w)$ equals the algebraic multiplicity of the eigenvalue $\lambda(w)$. (We could of course use the existence of the projection to give an alternative proof of Corollary 2.4.) Assuming also that $\dim \text{Ker}(P(w) - \lambda(w)\text{Id}_N) \equiv J$ in Y it follows that $\text{Ran } \Pi(w) = \text{Ker}(P(w) - \lambda(w)\text{Id}_N)$. Note that if w is fixed then the operator $\Pi(w)$ is idempotent and we have the direct sum

$$(5.1) \quad \mathbb{C}^N = \text{Ran } \Pi(w) \oplus \text{Ran}(\text{Id}_N - \Pi(w)).$$

Let V be the topological manifold

$$V = \{(w, z) : w \in Y, z \in \text{Ker}(P(w) - \lambda(w)\text{Id}_N)\},$$

and let $\pi : (w, z) \mapsto w$ be the projection. Then V can by means of Π be given the structure of a C^∞ complex vector bundle over Y . Indeed, it is clear that each fiber $V_w = \pi^{-1}(w) = \text{Ker}(P(w) - \lambda(w)\text{Id}_N)$ over w has a natural vector space structure induced from the one on \mathbb{C}^N . Since Y is open and $w \mapsto \Pi(w)$ is smooth, we can for each $w_\alpha \in Y$ find a neighborhood $U_\alpha \subset Y$ of w_α such that

$$w \in U_\alpha \implies \|\Pi(w) - \Pi(w_\alpha)\| < 1.$$

Choose orthonormal bases $\{e_{\alpha,1}, \dots, e_{\alpha,J}\}$ and $\{e_{\alpha,J+1}, \dots, e_{\alpha,N}\}$ of $\text{Ran } \Pi(w_\alpha)$ and $\text{Ker } \Pi(w_\alpha) = \text{Ran}(\text{Id}_N - \Pi(w_\alpha))$, respectively, so that $\Pi(w_\alpha)e_{\alpha,k} = e_{\alpha,k}$ for $1 \leq k \leq J$ and 0 otherwise. It is then easy to see that the C^∞ sections

$$(5.2) \quad U_\alpha \ni w \mapsto f_{\alpha,k}(w) = \Pi(w)e_{\alpha,k}, \quad 1 \leq k \leq J,$$

are linearly independent and therefore constitute a basis for each fiber V_w over U_α . (Note that the C^∞ sections

$$U_\alpha \ni w \mapsto f_{\alpha,k}(w) = (\text{Id}_N - \Pi(w))e_{\alpha,k}, \quad J+1 \leq k \leq N,$$

are also linearly independent.) This allows for the construction of the required local isomorphism ψ_α from $\pi^{-1}(U_\alpha)$ onto $U_\alpha \times \mathbb{C}^J$. Hence V is a C^∞ complex vector bundle of fiber dimension J , and (5.2) is a local frame for V over U_α . In fact, if $w \in U_\alpha \cap U_\beta$, then the columns of the transition matrix

$$g_{\alpha\beta} = \psi_\alpha \circ \psi_\beta^{-1} : U_\alpha \cap U_\beta \rightarrow GL(N, \mathbb{C})$$

are just the coordinates of the local frame over U_β in terms of the local frame over U_α . Since the local frames consist of C^∞ sections, this implies that $g_{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta)$. The same arguments show that the complimentary manifold

$$V' = \{(w, z) : w \in Y, z \in \text{Ker } \Pi(w)\}$$

is a C^∞ complex vector bundle over Y with fiber dimension $N - J$.

We shall need the fact that V and V' are trivial in the cases under consideration in Section 4.

Proposition 5.1. *Let $P \in \Psi_{\text{cl}}^m(X)$ be an $N \times N$ system with homogeneous principal symbol P_m . Let γ be a compact and smooth simple curve contained in the characteristic set $\Sigma(P_m)$ of P_m , and suppose that P is of principal type with constant characteristics near γ . Let $w \mapsto \lambda(w)$ be the unique section of eigenvalues of $P_m(w)$ vanishing along γ , and suppose that γ is a one dimensional bicharacteristic of λ with injective regular projection in $S^*(X)$. Then there exists a conic neighborhood Ω of γ and a positive number J such that*

$$(5.3) \quad V = \{(w, z) : w \in \Omega, z \in \text{Ker}(P_m(w) - \lambda(w)\text{Id}_N)\}$$

is a C^∞ complex vector bundle over Ω with fiber dimension J , where the fiber V_w over $w \in \Omega$ is given by $V_w = \text{Ker}(P_m(w) - \lambda(w)\text{Id}_N)$. Moreover, there is a local frame $\{z_1, \dots, z_J\}$ for V over Ω such that

$$z_k : \Omega \ni w \mapsto z_k(w) \in V_w, \quad 1 \leq k \leq J,$$

is homogeneous of degree 0 and an eigenvector of P_m with eigenvalue λ . Thus V is trivial. This local frame can be completed to a local frame for the the trivial vector bundle $F = \Omega \times \mathbb{C}^N$.

Proof. By assumption P_m has constant characteristics, so the characteristic equation

$$|P_m(w) - \lambda(w)\text{Id}_N| = 0$$

has the unique local solution $\lambda(w) \in C^\infty$ of multiplicity $J > 0$, where $\lambda(w)$ is the section of eigenvalues given in the statement of the proposition. Since P_m is of principal type, the geometric multiplicity $\dim \text{Ker}(P_m(w) - \lambda(w)\text{Id}_N) \equiv J$ in a neighborhood of γ by [4, Proposition 2.10]. If $\pi : T^*(X) \setminus 0 \rightarrow S^*(X)$ is the projection, it follows by homogeneity that we can find a neighborhood $\mathcal{V} \subset S^*(X)$ of $\pi \circ \gamma$ such that this still holds in the conic set $\pi^{-1}(\mathcal{V}) \subset T^*(X) \setminus 0$.

By introducing a Riemannian metric on X defining the unit cotangent bundle, we can as in the proof of Proposition 2.3 write $P_m(x, \xi) = |\xi|^m \pi^* p_s(x, \xi)$ and $\lambda(x, \xi) = |\xi|^m \pi^* \varrho_s(x, \xi)$ where p_s and ϱ_s are functions in $C^\infty(S^*(X))$ with values in \mathcal{L}_N and \mathbb{C} , respectively. In the neighborhood \mathcal{V} of $\pi \circ \gamma$ it follows by homogeneity that ϱ_s is the unique section of eigenvalues of p_s that vanishes along $\pi \circ \gamma$. In particular, $\pi \circ \gamma \subset \Sigma(p_s)$. With $v = \pi(w)$ for $w = (x, \xi) \in \pi^{-1}(\mathcal{V})$ it is also easy to see that

$$\text{Ker}(P_m(w) - \lambda(w)\text{Id}_N) = \text{Ker}(p_s(v) - \varrho_s(v)\text{Id}_N).$$

Thus, $\dim \text{Ker}(p_s(v) - \varrho_s(v)\text{Id}_N) \equiv J$ for $v \in \mathcal{V}$. By the discussion preceding the proposition it then follows that

$$\{(v, z) : v \in \mathcal{V}, z \in \text{Ker}(p_s(v) - \varrho_s(v)\text{Id}_N)\}$$

is a C^∞ complex vector bundle over \mathcal{V} . With the notation of the proposition it is clear that the pullback by π of the local frames constructed above yield local frames for V over open conic subsets of $T^*(X) \setminus 0$ whose union forms a conic neighborhood of γ , which proves the first part of the proposition. Similarly, if we can find C^∞ sections $v \mapsto z_k(v)$, $1 \leq k \leq J$, constituting a basis for $\text{Ker}(p_s(v) - \varrho_s(v)\text{Id}_N)$ for every $v \in \mathcal{V}$, then the collection $\{\pi^* z_k\}_{k=1}^J$ is a local frame for V over $\pi^{-1}(\mathcal{V})$ and the C^∞ sections $\pi^* z_k$ are homogeneous of degree 0. In particular, if $p_s(v)z_j(v) = \varrho_s(v)z_j(v)$ for $v = \pi(w) \in \mathcal{V}$ then

$$P_m(w)\pi^* z_j(w) = \lambda(w)\pi^* z_j(w).$$

If the local frame $\{z_1, \dots, z_J\}$ can be extended to a local frame for the trivial complex vector bundle $\mathcal{V} \times \mathbb{C}^N$, then the collection $\{\pi^* z_j\}_{j=1}^N$ has the required properties, thereby proving the proposition. Since γ has injective regular projection in $S^*(X)$, we can thus assume that γ is a curve on the cosphere bundle to begin with, while $P_m(w)$ and $\lambda(w)$ belong to $C^\infty(S^*(X))$. Thus (5.3) holds with Ω replaced by \mathcal{V} . Since γ is contractible by assumption, we find (after possibly shrinking \mathcal{V} if necessary) that V is trivial, see for example Corollary 4.8 in [5, Chapter 3]. If

$$V' = \{(w, z) : w \in \mathcal{V}, z \in \text{Ker } \Pi(w)\}$$

is the complimentary vector bundle over \mathcal{V} , the same reasoning shows that V' is trivial. Since the operator valued function $w \mapsto \text{Id}_N - \Pi(w)$ is the projection onto the null space V'_w of $\Pi(w)$, we have $\mathbb{C}^N = V_w \oplus V'_w$ for any $w \in \mathcal{V}$ by (5.1), so together the local frames for V and V' over \mathcal{V} give a local frame for the trivial vector bundle $F = \mathcal{V} \times \mathbb{C}^N$ over \mathcal{V} . This completes the proof. \square

We now prove that the local preparation result for systems given by Lemma 4.1 in [4] can be generalized to a neighborhood of a compact one dimensional bicharacteristic interval.

Lemma 5.2. *Let $P \in \Psi_{\text{cl}}^m(X)$ be an $N \times N$ system with principal symbol P_m . Let γ be a compact and smooth simple curve contained in the characteristic set $\Sigma(P_m)$ of P_m , and suppose that P is of principal type with constant characteristics near γ . Let $\lambda(w)$ be the unique section of eigenvalues of $P_m(w)$ vanishing along γ , and suppose that γ is a one dimensional bicharacteristic of λ with injective regular projection in $S^*(X)$. Then one can find $N \times N$ systems A and B in $\Psi_{\text{cl}}^0(X)$, non-characteristic in a conic neighborhood of γ , such that*

$$(5.4) \quad APB = \begin{pmatrix} \tilde{P}_{11} & 0 \\ 0 & \tilde{P}_{22} \end{pmatrix} \in \Psi_{\text{cl}}^m(X)$$

microlocally near γ . Moreover, \tilde{P}_{22} is elliptic, and we have $\sigma(\tilde{P}_{11}) = \lambda \text{Id}_J$ where the section of eigenvalues $\lambda(w) \in C^\infty$ of $P(w)$ is of principal type near γ .

Proof. First we note, as in the beginning of the proof of Proposition 5.1, that since P_m is of principal type with constant characteristics, the geometric multiplicity $\dim \text{Ker}(P_m(w) - \lambda(w)\text{Id}_N) \equiv J > 0$ in a conic neighborhood of γ , where $\lambda(w) \in C^\infty$ is the section of eigenvalues of multiplicity J given in the statement of the lemma. Moreover, [4, Proposition 2.10] also gives that $d\lambda \neq 0$ on γ , and since γ is a one dimensional bicharacteristic of λ with injective projection in $S^*(X)$ it follows that the composition of γ and the Hamilton vector field H_λ of λ does not have the radial direction. Indeed, as in the proof of [11, Theorem 26.4.12] we can for a suitably normalized parametrization $t \mapsto \gamma(t)$ of γ find a C^∞ function ϱ , homogeneous of degree 0, such that

$$0 \neq \gamma'(t) = \varrho(\gamma(t))H_\lambda(\gamma(t)) = H_{\text{Re } \varrho \lambda} \circ \gamma(t).$$

In particular, since $\lambda \circ \gamma = 0$ we find that γ is a bicharacteristic of the homogeneous function $\text{Re } \varrho \lambda$ such that $H_{\text{Re } \varrho \lambda} \neq 0$ along γ . Thus, if H_λ and the radial vector field are linearly dependent at some point on γ , then γ would just be a ray in the radial direction which is a contradiction since γ is assumed to have injective projection in $S^*(X)$. By homogeneity λ is then of principal type in a conic neighborhood of γ .

Now use Proposition 5.1 and an orthogonalization procedure to obtain a unitary $N \times N$ system E , homogeneous of degree 0 and non-characteristic in a conic neighborhood of γ , such that

$$E^* P_m E = \begin{pmatrix} \lambda(w)\text{Id}_J & P_{12} \\ 0 & P_{22} \end{pmatrix} = \tilde{P}_m$$

is the principal symbol of $A'PB'$ for any systems $A', B' \in \Psi_{\text{cl}}^0(X)$ having principal symbols E^* and E , respectively. Here $E^* = E^{-1}$ is the Hermitian adjoint of E . Inspecting the end of the proof of [4, Lemma 4.1] we find that the result now follows by essentially repeating the arguments found there. We omit the details. \square

6. THE PROOF OF THEOREM 4.6

Recall that we may assume that the systems P and Q given by Theorem 4.6 have the same order. Now note that if A and B are the elliptic systems given by Lemma 5.2, it follows as a special case of Proposition 4.3 that the range of Q is microlocally contained in the range of P at K if and only if the range of AQB is microlocally contained in the range of APB at K . Since A and B are elliptic, it is easy to see using the calculus that all the terms in the asymptotic expansion of the symbol of Q have vanishing Taylor coefficients if and only if the same holds for AQB . Note also that when γ is a minimal characteristic point, the normal form given by Lemma 5.2 is still valid near γ in view of [4, Lemma 4.1]. We can thus reduce the proof of Theorem 4.6 to the case when P has the form given by (5.4) and $\lambda = \lambda(w)$ is the unique eigenvalue of the principal symbol P_m of P satisfying $\lambda \circ \gamma = 0$. In view of Lemma A.1 in the appendix, we can use Proposition 4.3 together with [10, Theorem 21.3.6] or [11, Theorem 26.4.13] when γ is a characteristic point or a one dimensional bicharacteristic, respectively, to further reduce the proof to the case $Q, P \in \Psi_{\text{cl}}^1(\mathbb{R}^n)$, $\gamma(x_1) = (x_1, 0, \varepsilon_n) \in T^*(\mathbb{R}^n)$ for $x_1 \in I$, and

$$(6.1) \quad \lambda(x, \xi) = \xi_1 + if(x, \xi')$$

where f is real valued, homogeneous of degree 1 and independent of ξ_1 . Note that under these hypotheses, γ is still a minimal characteristic point or a minimal bicharacteristic interval of λ . Thus, in any neighborhood of γ one can find an interval in the x_1 direction where f changes sign from $-$ to $+$ for increasing x_1 . If γ is not reduced to a point then f vanishes of infinite order on γ by Proposition 3.4, and by Theorem 3.6 we can find a sequence $\{I_j\}_{j=1}^\infty$ of ϱ_j -minimal bicharacteristic intervals such that $\varrho_j \rightarrow 0$ and $I_j \rightarrow \gamma$ as $j \rightarrow \infty$. Note also that we still have

$$(6.2) \quad P = \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix}$$

where $\sigma(P_{11}) = \lambda \text{Id}_J$ and P_{22} is elliptic microlocally near $\gamma(I)$.

Let

$$(6.3) \quad Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$$

be the block form of Q corresponding to (6.2), so that for example Q_{12} is a $J \times (N - J)$ system. If P_{22}^{-1} is a microlocal parametrix of P_{22} near $\gamma(I)$, then

$$Q = P \cdot \begin{pmatrix} 0 & 0 \\ P_{22}^{-1}Q_{21} & P_{22}^{-1}Q_{22} \end{pmatrix} + \begin{pmatrix} Q_{11} & Q_{12} \\ 0 & 0 \end{pmatrix}$$

mod $\Psi^{-\infty}$ microlocally near $\gamma(I)$. We may assume that $J > 0$ since otherwise P is elliptic. Now let

$$(6.4) \quad \sigma_{Q_{11}} = q_1 + q_0 + \dots$$

be the total symbol of Q_{11} , where q_j is a $J \times J$ system, homogeneous of degree j . With λ given by (6.1) we have $\sigma(P_{11}) = 0$ and $|\partial_{\xi_1} \sigma(P_{11})| \neq 0$ at $\gamma(I)$, so in place of the Malgrange preparation theorem we can use (the transpose of) [1, Theorem A.4] to obtain

$$\sigma(Q_{11})(x, \xi) = \lambda(x, \xi) \text{Id}_J E_0(x, \xi) + R_1(x, \xi')$$

in a neighborhood of $\gamma(I)$ for some matrix valued smooth functions E_0 and R_1 , where R_1 is independent of ξ_1 . (Of course, since $\sigma(P_{11}) = \lambda \text{Id}_J$, the usual scalar Malgrange preparation theorem is actually sufficient.) Restricting to $\xi = 1$ and extending by homogeneity we can make E_0 and R_1 homogeneous of degree 0 and 1, respectively. We can repeat the argument for lower order terms and obtain $Q_{11} = P_{11} \circ E_{11} + R_{11}(x, D_{x'})$ where $E_{11} \in \Psi_{\text{cl}}^0(\mathbb{R}^n)$ and $R_{11} \in \Psi_{\text{cl}}^1(\mathbb{R}^n)$ are $J \times J$ systems, and the symbol of R_{11} is independent of ξ_1 . Doing the same for Q_{12} we get

$$Q = P \cdot \begin{pmatrix} E_{11} & E_{12} \\ P_{22}^{-1}Q_{21} & P_{22}^{-1}Q_{22} \end{pmatrix} + \begin{pmatrix} R_{11} & R_{12} \\ 0 & 0 \end{pmatrix}$$

mod $\Psi^{-\infty}$ microlocally near $\gamma(I)$. One easily checks that the range of

$$(6.5) \quad R(x, D_{x'}) = \begin{pmatrix} R_{11}(x, D_{x'}) & R_{12}(x, D_{x'}) \\ 0 & 0 \end{pmatrix}$$

is microlocally contained in the range of P near $\gamma(I)$. Hence Theorem 4.6 follows if we show that all the terms in the asymptotic expansion of the symbol of R have vanishing Taylor coefficients at $\gamma = \{(x_1, 0, \varepsilon_n) : x_1 \in I\}$. Note that when proving this we may assume that the lower order terms in the symbol of P_{11} are independent of ξ_1 . In fact, if $\sigma_{P_{11}} = \lambda \text{Id}_J + p_0 + \dots$ then [1, Theorem A.4] implies that

$$p_0(x, \xi) = a(x, \xi)(\xi_1 + if(x, \xi')) + b(x, \xi')$$

where a is homogeneous of degree -1 and b homogeneous of degree 0 , as demonstrated in the construction of the systems E and R above. The term of degree 0 in the symbol of $(\text{Id}_J - a(x, D))P_{11}$ is equal to $b(x, \xi')$. Repetition of the argument implies that there exists a $J \times J$ system of classical operators $A_{11}(x, D)$ of order -1 such that $(\text{Id}_J - A_{11})P_{11}$ has principal symbol $(\xi_1 + if(x, \xi'))\text{Id}_J$ and all lower order terms are independent of ξ_1 . If A is the $N \times N$ system

$$A(x, D) = \begin{pmatrix} \text{Id}_J - A_{11}(x, D) & 0 \\ 0 & \text{Id}_{N-J} \end{pmatrix}$$

then the microlocal property of pseudodifferential operators immediately implies that the range of AQ is microlocally contained in the range of AP at K . Hence, if there are systems E and R with

$$R = AQ - APE$$

such that all terms in the asymptotic expansion of the symbol of R have vanishing Taylor coefficients at $\gamma(I)$, then this also holds for the symbol of $Q - PE \equiv A^{-1}R \bmod \Psi^{-\infty}$, since the calculus gives that this property is preserved under composition with elliptic systems.

When γ is a minimal bicharacteristic interval, that is, when I is not reduced to a point, then we may assume that there exists a neighborhood of γ where the implication

$$(6.6) \quad f(x, \xi') = 0 \implies \partial f(x, \xi')/\partial x_1 \leq 0$$

holds. Indeed, if there is no such neighborhood, then as shown in [16] (see the discussion in connection with equation (2.19) there), we find that γ is just a point and there exists a point sequence $\{\gamma_j\}_{j=1}^\infty = \{(t_j, x'_j, 0, \xi'_j)\}_{j=1}^\infty$ such that $\gamma_j \rightarrow \gamma$ as $j \rightarrow \infty$, and

$$(6.7) \quad f(t_j, x'_j, \xi'_j) = 0, \quad \partial f(t_j, x'_j, \xi'_j)/\partial x_1 > 0$$

for each j . Note that (6.7) implies $\{\text{Re } \lambda, \text{Im } \lambda\}(\gamma_j) > 0$ and $\lambda(\gamma_j) = 0$ for each j since $\gamma_j = (t_j, x'_j, 0, \xi'_j)$. Thus, when γ is a minimal characteristic point we conclude that either there is a neighborhood where (6.6) holds, or we can find a sequence $\{\gamma_j\}_{j=1}^\infty$ with the properties given above. This will allow us to complete the proof of Theorem 4.6 using the following two results.

Theorem 6.1. *Let the $N \times N$ system P be given by (6.2) where P_{22} is elliptic, and suppose that in a conic neighborhood Ω of*

$$\Gamma' = \{(x_1, x', 0, \xi'), a \leq x_1 \leq b\} \subset T^*(\mathbb{R}^n) \setminus 0$$

the principal symbol of P_{11} has the form $\lambda(x, \xi)\text{Id}_J$ with

$$\lambda(x, \xi) = \xi_1 + if(x, \xi'),$$

where f is real valued and homogeneous of degree 1, while the lower order terms of the symbol of P_{11} are all independent of ξ_1 . Suppose also that (6.6) holds in Ω and that in any neighborhood of Γ' one can find an interval in the x_1 direction where f changes sign from $-$ to $+$ for increasing x_1 . Assume that if $b > a$ then f vanishes of infinite order on Γ' and there exists a $\varrho \geq 0$ such that for any $\varepsilon > \varrho$ one can find a neighborhood of

$$\Gamma'_\varepsilon = \{(x_1, x', 0, \xi'), a + \varepsilon \leq x_1 \leq b - \varepsilon\}$$

where f vanishes identically. Furthermore, let the $N \times N$ system $R(x, D_{x'})$ be given by (6.5) and suppose that in Ω the symbol of R is given by an asymptotic sum of homogeneous terms that are all independent of ξ_1 . If there exists a compactly based cone $K \subset T^*(\mathbb{R}^n) \setminus 0$ containing Ω such that the range of R is microlocally contained in the range of P at K , then all the terms in the asymptotic sum of the symbol of R have vanishing Taylor coefficients on Γ'_ϱ if $a < b$, and at Γ' if $a = b$.

Theorem 6.2. *Let the $N \times N$ system P be given by (6.2) where P_{22} is elliptic, and suppose that in a conic neighborhood Ω of*

$$\Gamma' = \{(0, \varepsilon_n)\} \subset T^*(\mathbb{R}^n) \setminus 0$$

P_{11} has the form $P_{11} = (D_1 + ix_1 D_n) \text{Id}_J$. Moreover, let the $N \times N$ system $R(x, D_{x'})$ be given by (6.5) and suppose that the symbol of R is given by an asymptotic sum of homogeneous terms that are all independent of ξ_1 . If there exists a compactly based cone $K \subset T^*(\mathbb{R}^n) \setminus 0$ containing Ω such that the range of R is microlocally contained in the range of P at K , then all the terms in the asymptotic sum of the symbol of R have vanishing Taylor coefficients on Γ' .

Postponing the proofs of these results we are now left with three cases:

i) γ is a minimal bicharacteristic interval. Then there is a neighborhood Ω of γ where (6.6) holds, and since $\text{Im } \lambda = f$ is homogeneous we may assume that Ω is conic. By Theorem 3.6 there exists a sequence $\{\Gamma_j\}_{j=1}^\infty$ of ϱ_j -minimal bicharacteristic intervals such that $\varrho_j \rightarrow 0$ and $\Gamma_j \rightarrow \gamma$ as $j \rightarrow \infty$. For sufficiently large j we have $\Gamma_j \subset \Omega$. If

$$\Gamma_j = \{(x_1, x'_j, 0, \xi'_j) : a_j \leq x_1 \leq b_j\}$$

then all the terms in the asymptotic sum of the symbol of R vanish of infinite order on

$$\Gamma_{\varrho_j} = \{(x_1, x'_j, 0, \xi'_j) : a_j + \varrho_j \leq x_1 \leq b_j - \varrho_j\}$$

by Theorem 6.1. Since $\Gamma_{\varrho_j} \rightarrow \gamma$ as $j \rightarrow \infty$, and all the terms in the asymptotic sum of the symbol of R are smooth functions, it follows that all the terms in the asymptotic sum of the symbol of R vanish of infinite order on γ , thus proving Theorem 4.6 in this case.

ii) γ is a minimal characteristic point and (6.6) holds. Then all the terms in the asymptotic sum of the symbol of R vanish of infinite order on γ by Theorem 6.1, so Theorem 4.6 follows.

iii) γ is a minimal characteristic point and (6.6) is false. Let γ_j be a fixed point in the sequence $\{\gamma_j\}_{j=1}^\infty$ satisfying (6.7). Since P is given by (6.2) and the principal symbol of P_{11} is just the scalar function λ times the identity matrix, we can then by conjugating as in the scalar case (see the proof of [11, Theorem 26.3.1]) show that P_{11} is microlocally conjugate to $(D_1 + ix_1 D_n) \text{Id}_J$, which allows us to prove Theorem 4.6 by an application of Theorem 6.2. We prove this by adapting the arguments in [3, p. 18], where it is shown to hold for systems of semiclassical operators. Note that we now forgo the previous preparation $Q = PE + R$ with R given by (6.5), with the intention of recreating it after having conjugated P .

Since γ_j is fixed, we can by choosing appropriate local coordinates use [10, Theorem 21.3.3] to find a canonical transformation χ and a smooth function μ such that $\chi(0, \varepsilon_n) = \gamma_j$ and $\chi^*(\mu\lambda) = \xi_1 + ix_1 \xi_n$ near $(0, \varepsilon_n)$. By [11, Theorem 26.3.1] together with Lemma A.1 in the appendix we can then find systems \tilde{A} and \tilde{B} of Fourier integral operators such that $\tilde{P} = \tilde{B}P\tilde{A}$ is still on a normal form of the type

(6.2) with $\sigma(\tilde{P}_{11}) = (\xi_1 + ix_1\xi_n)\text{Id}_J$ in a conic neighborhood Ω of $(0, \varepsilon_n)$ and \tilde{P}_{22} elliptic. Let therefore

$$\tilde{P}_{11} = \lambda(x, D)\text{Id}_J + F,$$

where $\lambda(x, \xi) = \xi_1 + ix_1\xi_n$ and $F \in \Psi_{\text{cl}}^0(\mathbb{R}^n)$ has a symbol with asymptotic expansion $\sigma_F(w) \sim \sum_{j \geq 0} F_{-j}(w)$. Here F_{-k} is a matrix valued function, homogeneous of degree $-k$. Let the systems $A, B \in \Psi_{\text{cl}}^0(\mathbb{R}^n)$ have symbols $\sigma_A \sim \sum_{j \geq 0} A_{-j}$ and $\sigma_B \sim \sum_{j \geq 0} B_{-j}$ with $A_0(w) \equiv B_0(w)$. Then the calculus gives

$$\tilde{P}_{11}A - B\lambda(x, D)\text{Id}_J = E \in \Psi_{\text{cl}}^0(\mathbb{R}^n),$$

where the system E has symbol $\sigma_E \sim \sum_{j \geq 0} E_{-j}$ and

$$E_{-k} = \lambda(A_{-k-1} - B_{-k-1}) + F_0A_{-k} + \partial_\xi \lambda D_x A_{-k} - \partial_\xi B_{-k} D_x \lambda + R_{-k}.$$

Here R_{-k} only depends on A_{-j}, B_{-j} for $j < k$ and $R_0 \equiv 0$. Using the fact that

$$\begin{aligned} \partial_\xi \lambda D_x A_{-k} - \partial_\xi B_{-k} D_x \lambda &= \frac{1}{2i} H_\lambda (A_{-k} + B_{-k}) \\ &+ \frac{1}{2i} \left((\partial_x \lambda) \partial_\xi (A_{-k} - B_{-k}) + (\partial_\xi \lambda) \partial_x (A_{-k} - B_{-k}) \right), \end{aligned}$$

where H_λ is the Hamilton vector field of λ , we can therefore write

$$E_{-k} = \frac{1}{2i} H_\lambda (A_{-k} + B_{-k}) + \lambda(A_{-k-1} - B_{-k-1}) + F_0A_{-k} + R_{-k},$$

where R_{-k} now also depends on the difference $A_{-k} - B_{-k}$ in addition to A_{-j}, B_{-j} for $j < k$. Note that since $A_0(w) \equiv B_0(w)$ we still have $R_0 \equiv 0$. Now we can choose A_0 so that $A_0 = \text{Id}_J$ on $V_0 = \{w : \text{Im } \lambda(w) = 0\}$ and $\frac{1}{2i} H_\lambda A_0 + F_0 A_0$ vanishes of infinite order on V_0 near $(0, \varepsilon_n)$. In fact, since $\{\text{Re } \lambda, \text{Im } \lambda\} \neq 0$ at $(0, \varepsilon_n)$, we find that $H_{\text{Re } \lambda}$ and $H_{\text{Im } \lambda}$ are linearly independent at $(0, \varepsilon_n)$, and that $H_{\text{Re } \lambda}$ is not tangent to V_0 at $(0, \varepsilon_n)$. In particular, V_0 is near $(0, \varepsilon_n)$ a hypersurface such that $H_{\text{Re } \lambda}$ is transversal to V_0 . Thus, the equation determines all derivatives of A_0 on V_0 , and we can use Borel's theorem to obtain a solution. Next, we set

$$B_{-1} - A_{-1} = \left(\frac{1}{2i} H_\lambda A_0 + F_0 A_0 \right) \lambda^{-1} \in C^\infty$$

and obtain $E_0 \equiv 0$. This also completely determines R_{-1} . Similarly, lower order terms are eliminated by making

$$\frac{1}{2i} H_\lambda (A_{-k} + B_{-k}) + F_0 A_{-k} + R_{-k}$$

vanish of infinite order on V_0 . Note that since only the difference $B_{-k} - A_{-k}$ was determined in the previous step, this equation can be solved for A_{-k} , which then also determines B_{-k} . Next, by choosing $B_{-k-1} - A_{-k-1}$ appropriately we obtain $E_{-k} \equiv 0$, and in the process we also completely determine R_{-k-1} . Since B is microlocally invertible near $(0, \varepsilon_n)$ by construction, we find that

$$B^{-1} \tilde{P}_{11} A \equiv \lambda(x, D) \text{Id}_J \pmod{\Psi^{-\infty}}$$

near $(0, \varepsilon_n)$, if B^{-1} is a properly supported microlocal parametrix of B . Since Definition 4.1 is invariant under this type of composition by the discussion in the first paragraph of this section, we can let A and B^{-1} be included in the systems \tilde{A} and \tilde{B} of Fourier integral operators already introduced, and repeat the arguments above to obtain

$$\tilde{B} Q \tilde{A} = \tilde{B} P \tilde{A} E + R(x, D_{x'}),$$

where R is of the form (6.5) in a neighborhood of $(0, \varepsilon_n)$ with range microlocally contained in the range of $\tilde{B}P\tilde{A}$ at some compactly based cone K' containing Ω , and E and R have classical symbols. Then all the terms in the asymptotic expansion of the symbol of R vanish of infinite order at $(0, \varepsilon_n)$ by Theorem 6.2. If appropriate systems \tilde{A}' and \tilde{B}' of Fourier integral operators are chosen as in the proof of Lemma A.1 in the appendix, that is,

$$WF(\tilde{A}'\tilde{B} - \text{Id}_N) \cap K = \emptyset, \quad WF(\tilde{A}\tilde{B}' - \text{Id}_N) \cap K = \emptyset,$$

then Lemma A.1 implies that all the terms in the asymptotic expansion of the symbol of

$$Q - P\tilde{A}E\tilde{B}' \equiv \tilde{A}'(\tilde{B}Q\tilde{A} - \tilde{B}P\tilde{A}E)\tilde{B}' = \tilde{A}'R(x, D_{x'})\tilde{B}' \mod \Psi^{-\infty}(K)$$

vanish of infinite order at γ_j . Note that $\tilde{A}'R(x, D_{x'})\tilde{B}'$ now has the block form (6.5) in a neighborhood of γ_j . However, this neighborhood does not necessarily contain γ and the symbol is no longer necessarily independent of ξ_1 .

We have now shown that for each j there exists an operator $E_j \in \Psi_{\text{cl}}^0(\mathbb{R}^n)$ such that all the terms in the asymptotic expansion of the symbol of $Q - PE_j$ have vanishing Taylor coefficients at γ_j . To construct the operator E in Theorem 4.6, we do the following. For each j , write E_j in block form corresponding to that of P as $E_j = (E_{k\ell, j})$, $k, \ell = 1, 2$, and for $k = \ell = 1$ denote the symbol of $E_{11, j}$ by

$$e^j(x, \xi) \sim \sum_{\ell=0}^{\infty} e_{-\ell}^j(x, \xi)$$

where $e_0^j(x, \xi)$ is the principal part, and $e_{-\ell}^j(x, \xi)$ is homogeneous of degree $-\ell$. With Q given by (6.3) and the symbol of Q_{11} given by (6.4), let $\sigma_{P_{11}} = p_1 + p_0 + \dots$ so that $p_1 = \lambda \text{Id}_J$ is the principal symbol of P_{11} . It then follows by Proposition A.5 in the appendix that there exists a matrix valued function $e_0 \in C^\infty(T^*(\mathbb{R}^n) \setminus 0, \mathcal{L}_J)$, homogeneous of degree 0, such that $q_1 - p_1 e_0$ has vanishing Taylor coefficients at γ .

This argument can be repeated for lower order terms. Indeed, the term of degree 0 in the symbol of $Q_{11} - P_{11}E_{11, j}$ is

$$\sigma_0(Q_{11} - P_{11}E_{11, j}) = \tilde{q}_j - p_1 e_{-1}^j,$$

where

$$\tilde{q}_j(x, \xi) = q_0(x, \xi) - p_0(x, \xi)e_0^j(x, \xi) - \sum_k \partial_{\xi_k} p_1(x, \xi) D_{x_k} e_0^j(x, \xi).$$

We can write

$$p_1(x, \xi)e_{-1}^j(x, \xi) = p_1(x, \xi/|\xi|)e_{-1}^j(x, \xi/|\xi|),$$

so that $\tilde{q}_j(x, \xi)$, $p_1(x, \xi/|\xi|)$ and $e_{-1}^j(x, \xi/|\xi|)$ are all homogeneous of degree 0. Since

$$\partial_x^\alpha \partial_\xi^\beta e_0(\gamma) = \lim_{j \rightarrow \infty} \partial_x^\alpha \partial_\xi^\beta e_0^j(\gamma_j)$$

it follows by Proposition A.5 in the appendix that there is a matrix valued function $g \in C^\infty(T^*(\mathbb{R}^n) \setminus 0, \mathcal{L}_J)$, homogeneous of degree 0, such that

$$q_0(x, \xi) - p_0(x, \xi)e_0(x, \xi) - \sum_k \partial_{\xi_k} p_1(x, \xi) D_{x_k} e_0(x, \xi) - p_1(x, \xi/|\xi|)g(x, \xi)$$

has vanishing Taylor coefficients at γ . Putting $e_{-1}(x, \xi) = |\xi|^{-1}g(x, \xi)$ we find that

$$\partial_x^\alpha \partial_\xi^\beta e_{-1}(\gamma) = \lim_{j \rightarrow \infty} \partial_x^\alpha \partial_\xi^\beta e_{-1}^j(\gamma_j),$$

and that

$$\sigma_0(Q_{11} - P_{11} \circ e_0(x, D) - P_{11} \circ e_{-1}(x, D))$$

has vanishing Taylor coefficients at γ . Continuing this way we successively obtain matrix valued functions $e_m(x, \xi) \in C^\infty(T^*(\mathbb{R}^n) \setminus 0, \mathcal{L}_J)$, homogeneous of degree m for $m \leq 0$, such that

$$\sigma_{Q_{11}} - \left(\sum_{m=0}^M e_{-m} \right) \sigma_{P_{11}} \quad \text{mod } S_{\text{cl}}^{-M}$$

has vanishing Taylor coefficients at γ . If we let E_{11} have symbol

$$\sigma_{E_{11}}(x, \xi) \sim \sum_{m=0}^{\infty} (1 - \phi(\xi)) e_{-m}(x, \xi)$$

with scalar $\phi \in C_0^\infty$ equal to 1 for ξ close to 0, then $E_{11} \in \Psi_{\text{cl}}^0(\mathbb{R}^n)$ and all terms in the asymptotic expansion of the symbol of $Q_{11} - P_{11}E_{11}$ have vanishing Taylor coefficients at γ . Given that P has the form (6.2), these arguments can be repeated to construct a $J \times (N - J)$ system E_{12} such that all terms in the asymptotic expansion of the symbol of $Q_{12} - P_{11}E_{12}$ have vanishing Taylor coefficients at γ . By substituting Proposition A.6 for Proposition A.5 throughout, these arguments also show that there is an $(N - J) \times J$ system E_{21} and an $(N - J) \times (N - J)$ system E_{22} such that all terms in the asymptotic expansion of the symbol of $Q_{2\ell} - P_{22}E_{2\ell}$ have vanishing Taylor coefficients at γ for $\ell = 1, 2$. Then $E = (E_{k\ell})$ has the required properties.

It remains to prove Theorems 6.1 and 6.2. Since the system R in both results share some properties, we begin with a general discussion. First, as in the scalar case we note that in view of the calculus it suffices to prove the theorems for the adjoint

$$(6.8) \quad R^*(x, D_{x'}) = \begin{pmatrix} R_{11} & \dots & R_{1J} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ R_{N1} & \dots & R_{NJ} & 0 & \dots & 0 \end{pmatrix}$$

of R . Let therefore the symbol of R^* have the asymptotic expansion

$$(6.9) \quad \sigma_{R^*} \sim \sum_{j=-1}^{\infty} r_{-j},$$

where r_{-j} is the homogeneous matrix of degree $-j$ in the asymptotic sum of the symbol of R^* . Regarding the Taylor coefficients of r_{-j} as matrices, we can for any point (x_0, ξ_0) belonging to Γ' then use the ordering $>_t$ given by [16, Definition 3.2] to find the first nonzero matrix $\mathcal{R}_0 = r_{-j_0(\alpha_0)}^{(\beta_0)}(x_0, \xi_0)$ with respect to $>_t$. If $j_0 + |\alpha_0| + |\beta_0| = m_0$ for some number m_0 , then in particular all matrices $r_{-j(\alpha)}^{(\beta)}(x_0, \xi_0)$ equal the zero matrix for $j + |\alpha| + |\beta| < m_0$. Since the ordering will not appear explicitly in the proof we refrain from describing it further. We will assume that we have a nonzero entry in the first row and the first column in the matrix \mathcal{R}_0 ,

but this will only affect the construction below in an obvious manner, so it is of no importance.

Proof of Theorem 6.1. We shall prove the theorem by contradiction, arguing that if it is false, then Lemma 4.2 does not hold. This will be accomplished by constructing approximate solutions to the equation $P^*v = 0$ concentrated near Γ' in such a way that the proof reduces to the scalar case. Note that the symbol of R^* is independent of ξ_1 , and that R^* acting on a vector $v \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N)$ only depends on the first J coordinates of v . Hence we can let the approximate solutions be vectors in $\mathbb{C}^J \times \{0\} \subset \mathbb{C}^N$. We shall let each component be an approximate solution to a scalar problem of the same kind, constructed as in [16, Section 4].

To simplify notation, we shall in what follows write t instead of x_1 and x instead of $x' = (x_2, \dots, x_n)$, and we may without loss of generality assume that Γ' is given by

$$\Gamma' = \{(t, 0, 0, \xi^0) : a \leq t \leq b\},$$

where $\xi^0 = (0, \dots, 0, 1) \in \mathbb{R}^{n-1}$. Let K and Ω be the cones given by Theorem 6.1. Given any positive integer M we can by [11, Lemma 26.4.14] find a curve $t \mapsto (t, y(t), 0, \eta(t))$ as close to Γ' as desired, and functions w_0 and w_α such that

$$(6.10) \quad w(t, x) = w_0(t) + \langle x - y(t), \eta(t) \rangle + \sum_{2 \leq |\alpha| \leq M} w_\alpha(t) (x - y(t))^\alpha / |\alpha|!$$

is a formal solution to the eiconal equation

$$(6.11) \quad \partial w / \partial t - i f(t, x, \partial w / \partial x) = 0$$

with an error of order $\mathcal{O}(|x - y(t)|^{M+1})$ in a neighborhood Y of

$$(6.12) \quad \{(t, 0) : a \leq t \leq b\} \subset \mathbb{R}^n,$$

such that $\text{Im } w > 0$ in Y except on a compact non-empty subset T of the curve $x = y(t)$, while $w = 0$ on T . By part (i) of [11, Lemma 26.4.14] we can choose w so that

$$\Gamma_0 = \{(t, x, \partial w(t, x) / \partial t, \partial w(t, x) / \partial x) : (t, x) \in T\}$$

is contained in Ω , which is done to ensure that if A is a given system of pseudo-differential operators with wave front set contained in the complement of K , then $WF(A)$ does not meet the cone generated by Γ_0 . Note also that the functions w_α can be chosen so that for $|\alpha| = 2$ we have that the matrix $\text{Im } w_{jk} - \delta_{jk}/2$ is positive definite, where δ_{jk} is the Kronecker delta. If Γ' is a point we can thus obtain a sequence $\{\gamma_j\}_{j=1}^\infty$ of curves

$$\gamma_j(t) = (t, y_j(t), 0, \eta_j(t)), \quad a'_j \leq t \leq b'_j,$$

approaching Γ' together with solutions w_j to (6.11) which implies that at $t = c'_j$ we have

$$(c'_j, y_j(c'_j), 0, \eta_j(c'_j)) \rightarrow \Gamma' \quad \text{as } j \rightarrow \infty$$

in $T^*(\mathbb{R}^n) \setminus 0$, where c'_j is the point where $\text{Re } w_{0j} = \text{Im } w_{0j} = 0$. Similarly, if Γ' is an interval and $\varrho \geq 0$ is the number given by Theorem 6.1, then for any point ω in the interior of Γ'_ϱ we can use [16, Lemma 4.1] in place of [11, Lemma 26.4.14] to obtain a sequence $\{\gamma_j\}_{j=1}^\infty$ of curves approaching Γ' and a sequence $\{w_{0j}\}_{j=1}^\infty$ of functions such that for each j there exists a point $\omega_j \in \gamma_j$ with $\omega_j = \gamma_j(t_j)$ which can be chosen so that $\text{Re } w_{0j}(t_j) = \text{Im } w_{0j}(t_j) = 0$ and $\omega_j \rightarrow \omega$ as $j \rightarrow \infty$. If all the terms in the asymptotic sum of the symbol of R^* have vanishing Taylor coefficients

at ω_j , or at $(c'_j, y_j(c'_j), 0, \eta_j(c'_j))$ when Γ' is a point, then Theorem 6.1 will follow by continuity. In what follows we will suppress the index j to simplify notation, and we will show that all the terms in the asymptotic sum of the symbol of R^* have vanishing Taylor coefficients at one of these points, denoted henceforth by ω_0 , with $\omega_0 = \gamma(t_0)$ for some curve

$$(6.13) \quad t \mapsto \gamma(t) = (t, y(t), 0, \eta(t))$$

with the properties given above.

So suppose this is false, and let R^* be given by (6.8). Let M be a large positive integer to be determined later, and let w be of the form (6.10), corresponding to the curve $t \mapsto \gamma(t)$ containing ω_0 , such that w is an approximate solution to (6.11) with an error of order $\mathcal{O}(|x - y(t)|^{M+1})$ in a neighborhood Y of (6.12). Let N_0 be the integer given by Definition 4.1, and for $1 \leq k \leq J$ let $v_{k,\tau} \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$ be an approximate solution of the form

$$(6.14) \quad v_{k,\tau}(t, x) = e^{i\tau w(t,x)} \sum_{m=0}^M \phi_{k,m}(t, x) \tau^{-m}.$$

Here the amplitude functions $\phi_{k,m} \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$ are to be determined shortly. Let

$$V_\tau = \tau^{N_0+n}(v_{1,\tau}, \dots, v_{J,\tau}, 0) \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N).$$

Note that the $v_{k,\tau}$'s are approximate solutions of the same type as those in [11, Section 26.4]. Taking the additional factor τ^{N_0+n} in V_τ into account, it therefore follows by [11, Lemma 26.4.15] that we have

$$(6.15) \quad \|V_\tau\|_{(-N_0-n-\kappa)} \leq C\tau^{-\kappa}, \quad \tau > 1,$$

$$(6.16) \quad \|AV_\tau\|_{(0)} \leq C\tau^{-\kappa}, \quad \tau > 1,$$

for any $\kappa > 0$ if A is a pseudodifferential operator with wave front set disjoint from the cone generated by

$$(6.17) \quad \{(t, x, w'(t, x)) : x \in \bigcup_{k,m} \text{supp } \phi_{k,m}, \text{Im } w(t, x) = 0\}.$$

If ν is the number given by Lemma 4.2 and κ is any positive number, then our goal is to choose the amplitude functions so that

$$(6.18) \quad \|P^*V_\tau\|_{(\nu)} \leq C\tau^{-\kappa}$$

if the number M given by (6.14) is sufficiently large. Note that this estimate is not affected if the amplitude functions $\phi_{k,m}$ are multiplied with a cutoff function in $C_0^\infty(Y, \mathbb{R})$ which is 1 in a neighborhood of the compact set where $\text{Im } w = 0$. Since the $\phi_{k,m}$'s will be irrelevant outside Y for large τ by construction, we can in this way choose them to be supported in Y so that $V_\tau \in C_0^\infty(Y, \mathbb{C}^N)$. Now,

$$(6.19) \quad P^*V_\tau = \begin{pmatrix} \tau^{N_0+n} P_{11}^* t(v_{1,\tau}, \dots, v_{J,\tau}) \\ 0 \end{pmatrix},$$

and by the assumptions of Theorem 6.1 we can write $P_{11}^* = (D_t - if(t, x, D_x))\text{Id}_J + F_0(t, x, D_x)$ for some system $F_0 \in \Psi_{\text{cl}}^0(\mathbb{R}^n)$ with symbol depending on t, x and ξ . Since we have $\text{Im } w(t, x) > 0$ everywhere except at some points belonging to the curve $(t, x) = (t, y(t))$ where $w'_x(t, y(t)) = \eta(t)$ and $\text{Im } w''$ is positive definite by construction, we can use [11, Lemma 26.4.16] to obtain a formula for how P^* acts

on V_τ . In view of the discussion following that result, we find that since f is homogeneous of degree 1 we have

$$f(t, x, D_x)(e^{i\tau w} \phi_{k,m}) = e^{i\tau w} \sum_{|\alpha| \leq M} f^{(\alpha)}(t, x, \tau w'_x) D_x^\alpha \phi_{k,m} + \mathcal{O}(\tau^{(1-M)/2})$$

for $1 \leq k \leq J$. Here $f(t, x, \xi)$ is not defined for complex ξ , but since $w'_x(t, y(t)) = \eta(t)$, the expression $f^{(\alpha)}(t, x, \tau w'_x)$ is given meaning if it for each multi-index $\alpha \in \mathbb{N}^{n-1}$ is replaced by a finite Taylor expansion at $\tau \eta(t)$ representing the value at $\tau w'_x(t, x)$.

Now recall that w is an approximate solution to (6.11) with an error of order $\mathcal{O}(|x - y(t)|^{M+1})$. Since a function $\chi(t, x)e^{i\tau w}$ can be estimated by $\tau^{-\ell/2}$ if χ vanishes of order ℓ when $x = y(t)$ it follows that

$$e^{i\tau w}(\tau w'_t - i\tau f(t, x, w'_x))\phi_{k,m} = \mathcal{O}(\tau^{(1-M)/2}).$$

Recalling the definition of $v_{k,\tau}$ and using the homogeneity of f we thus obtain

$$(6.20) \quad (D_t - i f(t, x, D))v_{k,\tau} = e^{i\tau w} \sum_{m=0}^M \tau^{-m} \psi_{k,m} + \mathcal{O}(\tau^{(1-M)/2})$$

where

$$\psi_{k,m} = D_t \phi_{k,m} - \sum_{1 \leq |\alpha| \leq M} i \tau^{1-|\alpha|} f^{(\alpha)}(t, x, w'_x) D_x^\alpha \phi_{k,m}.$$

If $\sigma_{F_0}(t, x, \xi) \sim \sum_{j=0}^\infty f_{-j}(t, x, \xi)$ where $f_{-j} = (f_{k\ell, -j})$ are $J \times J$ matrices homogeneous of degree $-j$, then we can use the homogeneity of f_{-j} and apply [11, Lemma 26.4.16] to obtain

$$F_0^t(v_{1,\tau}, \dots, v_{J,\tau}) = e^{i\tau w} A_\tau(t, x) + \mathcal{O}(\tau^{(1-M)/2})$$

where

$$(6.21) \quad A_\tau(t, x) = \begin{pmatrix} \sum_{j,\ell,m,\alpha} \tau^{-j-|\alpha|-m} f_{1\ell,-j}^{(\alpha)}(t, x, w'_x) D_x^\alpha \phi_{\ell,m} / \alpha! \\ \vdots \\ \sum_{j,\ell,m,\alpha} \tau^{-j-|\alpha|-m} f_{J\ell,-j}^{(\alpha)}(t, x, w'_x) D_x^\alpha \phi_{\ell,m} / \alpha! \end{pmatrix}$$

and the sum is taken over $1 \leq \ell \leq J$ and all $0 \leq j \leq M'$, $0 \leq m \leq M$, $|\alpha| < M - 1 - 2j$ for some sufficiently large M' (see equation (4.21) in [16]). In (6.21) we should replace $f_{k\ell,-j}^{(\alpha)}(t, x, w'_x)$ by a Taylor expansion at $\eta(t)$ as above. Hence equations (6.20)–(6.21) imply that

$$P_{11}^{*t}(v_{1,\tau}, \dots, v_{J,\tau}) = e^{i\tau w} \begin{pmatrix} \sum_{m=0}^M \tau^{-m} \Psi_{1,m} \\ \vdots \\ \sum_{m=0}^M \tau^{-m} \Psi_{J,m} \end{pmatrix} + \mathcal{O}(\tau^{(1-M)/2}),$$

where

$$\Psi_{k,m} = D_t \phi_{k,m} - \sum_{|\alpha|=1} i f^{(\alpha)}(t, x, w'_x) D_x^\alpha \phi_{k,m} + \sum_{\ell=1}^J f_{k\ell,0}(t, x, w'_x) \phi_{\ell,m} + R_{k,m}$$

with $R_{k,0} = 0$ for $1 \leq k \leq J$ and $R_{k,m}$ determined by $\phi_{\ell,0}, \dots, \phi_{\ell,m-1}$, $1 \leq \ell \leq J$, for $m > 0$. Set

$$\phi_{k,0}(t, x) = \sum_{|\alpha| < M} \phi_{k,0\alpha}(t) (x - y(t))^\alpha$$

where $y(t)$ is the x coordinate of the curve $t \mapsto \gamma(t)$ in (6.13) containing the point ω_0 . Then $\Psi_{k,0}(t, x) = \mathcal{O}((x - y(t))^M)$ for $1 \leq k \leq J$ if $\phi_{k,0\alpha}$ satisfy a certain linear system of ordinary differential equations

$$(6.22) \quad D_t \phi_{k,0\alpha} + \sum_{\substack{1 \leq \ell \leq J \\ |\beta| < M}} a_{k\ell, \alpha\beta} \phi_{\ell,0\beta} = 0.$$

Given any non-negative integer $m_0 < M$, these equations may be solved so that, for example, $D_x^\alpha \phi_{k,0} = 0$ at $(t_0, y(t_0))$ for all $|\alpha| \leq m_0$ and $2 \leq k \leq J$, while $D_x^\alpha \phi_{1,0}(t_0, y(t_0)) = 0$ for all $|\alpha| \leq m_0$ except for one index α_0 with $|\alpha_0| = m_0$. We may in the same way successively choose $\phi_{k,m}$ for $1 \leq k \leq J$ so that

$$\Psi_{k,m}(t, x) = \mathcal{O}((x - y(t))^{M-2m}) \quad \text{when } m < M/2.$$

Using again the fact that a function of the form $\chi(t, x)e^{i\tau w}$ can be estimated by $\tau^{-\ell/2}$ if χ vanishes of order ℓ when $x = y(t)$, it follows that if M is chosen so that $(1 - M)/2 \leq -N_0 - n - \nu - \kappa$, then we obtain $P^*V_\tau = \mathcal{O}(\tau^{-\nu-\kappa})$ in view of (6.19). By the discussion in [11, p. 110] we conclude that for any integer κ we can find a constant C such that (6.18) holds if only $M = M(\kappa)$ is chosen sufficiently large.

Recall that R^* is given by (6.8), and let the symbol of R^* have the asymptotic expansion given by (6.9). Since we will prove Theorem 6.1 by contradiction, suppose that $\mathcal{R}_0 = r_{-j_0(\alpha_0)}^{(\beta_0)}(\omega_0)$ is the first nonzero matrix with respect to the ordering $>_t$ given by [16, Definition 3.2], where

$$(6.23) \quad j_0 + |\alpha_0| + |\beta_0| = m_0.$$

Here $\omega_0 = (t_0, y(t_0), 0, \eta(t_0))$. As mentioned above we will assume that we have a nonzero entry in the first row and the first column in the matrix \mathcal{R}_0 . Now let $H \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$, and define $h_\tau : \mathbb{R}^n \rightarrow \mathbb{C}$ by

$$h_\tau(t, x) = H(\tau(t - t_0), \tau(x - y(t))).$$

With $\mathcal{H}_\tau : \mathbb{R}^n \rightarrow \mathbb{C}^N$ given by $\mathcal{H}_\tau = \tau^{-N_0}(h_\tau, 0)$ it follows by [16, Proposition 4.3] that for $\tau \geq 1$ we have $\mathcal{H}_\tau \in H_{(N_0)}(\mathbb{R}^n, \mathbb{C}^N)$ and $\|\mathcal{H}_\tau\|_{(N_0)} \leq C$ where the constant depends on H but not on τ . In fact, the proof shows that $\|\mathcal{H}_\tau\|_{(N_0)} \leq C\tau^{-n/2}$ for $\tau \geq 1$ but this is not needed. (If we have a nonzero entry on the i :th row in the matrix \mathcal{R}_0 , then choose \mathcal{H}_τ as above with h_τ on the i :th coordinate.) Then

$$(6.24) \quad (R^*V_\tau, \overline{\mathcal{H}_\tau})_{L^2(\mathbb{R}^n, \mathbb{C}^N)} = \sum_{k=1}^J \tau^n (R_{1k} v_{k,\tau}, \overline{h_\tau}),$$

where (\cdot, \cdot) denotes the usual scalar product on $L^2(\mathbb{R}^n, \mathbb{C})$, and by Lemma 4.2 applied to the system R together with equations (6.15), (6.16) and (6.18), the left-hand side can be estimated by $C_\kappa \tau^{-\kappa}$ for any κ . As in the proof of [16, Theorem 2.21] we want to determine the limit of

$$\tau^{m_0} (R^*V_\tau, \overline{\mathcal{H}_\tau})_{L^2(\mathbb{R}^n, \mathbb{C}^N)}$$

as $\tau \rightarrow \infty$ with m_0 given by (6.23), and show that if the terms of the symbol of R^* do not all vanish of infinite order at ω_0 then H can be chosen so that this limit is nonzero, which is the contradiction that proves the theorem. For each integral in the right-hand side of (6.24) we can use [11, Lemma 26.4.16] and homogeneity to obtain an auxiliary formula for (6.24) as an asymptotic series in τ , where the coefficients consist among other things of derivatives in x of the amplitude functions

$\phi_{k,m}$. After the change of variables $(\tau(t-t_0), \tau(x-y(t))) \mapsto (t, x)$ we Taylor expand each term in the asymptotic sum to sufficiently high order, and then sort the result in declining homogeneity degree in τ (see equations (4.21)-(4.23) together with (4.33) in [16], and note that there, t_0 is assumed to be 0). If $\pi : T^*(\mathbb{R}^n) \rightarrow \mathbb{R}^n$ is the projection onto the base manifold, and we for $2 \leq k \leq J$ choose $\phi_{k,0}$ to have vanishing Taylor coefficients with respect to the x variable at $\pi(\omega_0) = (t_0, y(t_0))$ of sufficiently high order, then in view of equation (4.34) in [16] we see that the only contribution in (6.24) will come from $(Q_{11}v_{1,\tau}, \overline{h_\tau})$. (If on the i :th row we have a nonzero entry in the j :th column in \mathcal{R}_0 , choose $\phi_{k,0}$ as above for all $k \neq j$.) Since this reduces the situation to the scalar case, the theorem follows by repeating the proof of [16, Theorem 2.21]. \square

We now prove Theorem 6.2 using the same strategy as the one used to prove Theorem 6.1.

Proof of Theorem 6.2. We first construct approximate solutions to the equation $P^*v = 0$ concentrated near $I' = \{(0, \varepsilon_n)\}$. As in the proof of Theorem 6.1 we can let the approximate solutions be vectors in $\mathbb{C}^J \times \{0\} \subset \mathbb{C}^N$, and we will again let each component be an approximate solution to a scalar problem of the same kind, constructed this time as in [16, Section 3]. Thus, for $1 \leq k \leq J$ let $v_{k,\tau} \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$ be an approximate solution of the form

$$v_{k,\tau}(x) = \phi_k(x)e^{i\tau w(x)}$$

where

$$(6.25) \quad w(x) = x_n + i(x_1^2 + x_2^2 + \dots + x_{n-1}^2 + (x_n + ix_1^2/2)^2)/2$$

is a solution to $P^*w = 0$ and $\phi_k \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$. By the Cauchy-Kovalevsky theorem we can solve $D_1\phi_k - ix_1D_n\phi_k = 0$ in a neighborhood of 0 for any analytic initial data $\phi_k(0, x') = f_k(x') \in C^\omega(\mathbb{R}^{n-1}, \mathbb{C})$; in particular we are free to specify the Taylor coefficients of $f_k(x')$ at $x' = 0$. For $1 \leq k \leq J$ we take ϕ_k to be such a solution. If need be we can reduce the support of each ϕ_k by multiplying by a smooth cutoff function χ where χ is equal to 1 in some smaller neighborhood of 0 so that $\chi\phi_k$ solves the equation there. We assume this to be done and note that if the support of each ϕ_k is small enough then

$$\operatorname{Im} w(x) \geq |x|^2/4, \quad x \in \bigcup_k \operatorname{supp} \phi_k.$$

Since

$$d\operatorname{Re} w(x) = -x_1x_ndx_1 + (1 - x_1^2/2)dx_n$$

we may similarly assume that $d\operatorname{Re} w(x) \neq 0$ in $\operatorname{supp} \phi_k$, $1 \leq k \leq J$. If $V_\tau = \tau^{N_0+n}(v_{1,\tau}, \dots, v_{J,\tau}, 0) \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N)$, then by [11, Lemma 26.4.15] it follows that for any $\kappa > 0$ there is a constant C such that

$$(6.26) \quad \|V_\tau\|_{(-N_0-n-\kappa)} \leq C\tau^{-\kappa}, \quad \tau > 1,$$

$$(6.27) \quad \|AV_\tau\|_{(0)} \leq C\tau^{-\kappa}, \quad \tau > 1,$$

if A is a pseudodifferential operator with wave front set disjoint from the cone generated by

$$\{(x, w'(x)) : x \in \bigcup_k \operatorname{supp} \phi_k, \operatorname{Im} w(x) = 0\}.$$

Since

$$P^*V_\tau = {}^t\left((D_1 - ix_1D_n)(e^{i\tau w}\phi_1), \dots, (D_1 - ix_1D_n)(e^{i\tau w}\phi_J), 0, \dots, 0\right)$$

by construction, it follows that

$$(6.28) \quad \tau^m \|P^*V_\tau\|_{(\nu)} \rightarrow 0 \quad \text{as } \tau \rightarrow \infty$$

for any positive integers m and ν by [16, Lemma 3.1].

Now note that if we write t instead of x_1 and x instead of x' , then the solution w to $(D_1 - ix_1D_n)w = 0$ given by (6.25) takes the form

$$(6.29) \quad w(t, x) = i(t^2 - t^4/4)/2 + \langle x, (1 - t^2/2)\xi^0 \rangle + i|x|^2/2,$$

where as usual $\xi^0 = (0, \dots, 1) \in \mathbb{R}^{n-1}$. Comparing this to the solution of the eiconal equation given by (6.10), we see that (6.29) is the special case $w_0(t) = i(t^2 - t^4/4)/2$, $y(t) \equiv 0$, $\eta(t) = (1 - t^2/2)\xi^0$ and $w_\alpha(t) \equiv 0$ for $|\alpha| \geq 3$, $w_{jk}(t) = i\delta_{jk}$ where δ_{jk} is the Kronecker δ . Thus, $t \mapsto (t, y(t), 0, \eta(t))$ is a curve through the point Γ' . Having established the estimates (6.26)–(6.28), Theorem 6.2 therefore follows if we repeat the end of the proof of Theorem 6.1. We omit the details. \square

In view of the construction of approximate solutions to $P^*v = 0$ in the proof of Theorem 6.1, we can now give a short proof of Theorem 4.4.

Proof of Theorem 4.4. Let K be the cone generated by $\gamma(I)$ and recall that we only have to verify the theorem when $\gamma(I)$ is a minimal bicharacteristic interval, that is, when case (b) holds. In view of Proposition 4.3 with $Q = \text{Id}_N$ we may assume that P has the block form given Lemma 5.2, with the principal symbol of the $J \times J$ system P_{11} satisfying $\sigma(P_{11})(w) = \lambda(w)\text{Id}_J$ where $\lambda(w)$ is the section of eigenvalues of P given by Theorem 4.4. In fact, since the systems A and B in Lemma 5.2 are homogeneous and non-characteristic in a neighborhood of $\gamma(I)$, we can find a microlocal parametrix E of $AQB = AB$ such that

$$WF(EAB - \text{Id}_N) \cap K = WF(ABE - \text{Id}_N) \cap K = \emptyset.$$

Applying Proposition 4.3 shows that P is solvable at K if and only if the range of AB is microlocally contained in the range of APB at K , and using the existence of E it is easy to see that the latter holds if and only if APB is solvable at K . (Alternatively, the proof of [11, Proposition 26.4.4] immediately generalizes to a proof for a corresponding result for square systems, so this could be used in place of Proposition 4.3.) Keeping this observation in mind, we can in view of Definition 3.3 then use Lemma A.1 in the appendix, again with $Q = \text{Id}_N$, to further reduce the proof to the case when $P \in \Psi_{\text{cl}}^1(\mathbb{R}^n)$, $\lambda(x, \xi) = \xi_1 + if(x, \xi')$ and

$$\gamma(I) = \{(x_1, 0, \varepsilon_n) : x_1 \in I\},$$

where f is real valued, homogeneous of degree 1 and independent of ξ_1 . Since the normal form of $\lambda(x, \xi)$ is only valid in a neighborhood of $\{(x_1, 0, \varepsilon_n) : x_1 \in I\}$ we actually have to use a pseudodifferential cutoff for this to hold, but this can be accomplished by adapting the arguments in [11, pp. 107–108].

Note that γ is a minimal bicharacteristic interval of $\lambda(w)$, so in every neighborhood of $\gamma(I)$ there is a bicharacteristics of $\text{Re } \lambda = \xi_1$ along which f changes sign from $-$ to $+$, and f vanishes of infinite order on $\gamma(I)$. Since we are assuming that $|I| > 0$ there is a neighborhood of $\gamma(I)$ where (6.6) holds. This is all that is required for us to repeat the construction of the approximate solutions to $P_{11}^*v = 0$ from the proof of Theorem 6.1, so let $V_\tau = (v_{1,\tau}, \dots, v_{J,\tau}, 0) \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N)$ be the

corresponding approximate solution to $P^*V = 0$. Assume to reach a contradiction that P is solvable at the cone K generated by $\gamma(I)$ and let N_0 be the integer given by Definition 4.1 with $Q = \text{Id}_N$. If A is the system given by Lemma 4.2 such that $WF(A) \cap K = \emptyset$, concentrate V_τ so close to γ so that $WF(A)$ does not meet the cone generated by (6.17). Note that V_τ differs from the approximate solutions in the proof of Theorem 6.1 by a factor of τ^{-N_0-n} . In any case, equations (6.15), (6.16) and (6.18) imply that V_τ can be constructed so that the right-hand side of (4.2) is bounded by $C\tau^{-\kappa}$ for any κ if $\tau > 1$. Finally, by the discussion following (6.22) we can choose at least one of the amplitude functions $\phi_{k,0}$ in the definition of the $v_{k,\tau}$'s to be non-vanishing at an appropriately chosen point, which by [11, Lemma 26.4.15] implies that $\|V_\tau\|_{(-N_0)} \geq c\tau^{-n/2-N_0}$ for some $c > 0$. Applying Lemma 4.2 with $Q = \text{Id}_N$ we obtain a contradiction, which completes the proof. \square

APPENDIX A.

Here we prove a few results used in the main text, related to how the property that all terms in the asymptotic expansion of the total symbol have vanishing Taylor coefficients is affected by various operations. Some of these results are straightforward generalizations of the corresponding results for the scalar case, see [16, Appendix A].

Lemma A.1. *Suppose X and Y are two C^∞ manifolds of the same dimension n . Let $K \subset T^*(X) \setminus 0$ and $K' \subset T^*(Y) \setminus 0$ be compactly based cones and let χ be a homogeneous symplectomorphism from a conic neighborhood of K' to one of K such that $\chi(K') = K$, and let Γ be the graph of χ . Let $P \in \Psi_{\text{cl}}^m(Y)$ be an $N \times N$ system of properly supported classical pseudodifferential operators in Y of the form*

$$(A.1) \quad P = \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix}$$

where the principal symbol of the $J \times J$ system P_{11} is given by $\sigma(P_{11}) = \lambda \text{Id}_J$ for some scalar function $\lambda \in C^\infty(T^*(Y) \setminus 0)$, homogeneous of degree m , and P_{22} is an $(N - J) \times (N - J)$ system, elliptic in a conic neighborhood of K' . Suppose that there exists a function $0 \neq q \in C^\infty(T^*(Y) \setminus 0)$ such that

$$(\chi^{-1})^*(q\lambda) = \xi_1 + if(x, \xi').$$

Then one can find $N \times N$ systems $A \in I_{\text{cl}}^{1-m}(X \times Y, \Gamma')$ and $B \in I_{\text{cl}}^0(Y \times X, (\Gamma^{-1})')$ of properly supported Fourier integral operators such that

- (i) A and B are non-characteristic at the restriction of the graphs of χ and χ^{-1} to K' and to K respectively, while $WF'(A)$ and $WF'(B)$ are contained in small conic neighborhoods,
- (ii) $APB \in \Psi_{\text{cl}}^1(X)$ has the form (A.1) with P_{jj} replaced by \tilde{P}_{jj} for $j = 1, 2$, where $\sigma(\tilde{P}_{11}) = (\xi_1 + if(x, \xi'))\text{Id}_J$ and \tilde{P}_{22} is elliptic in a conic neighborhood of K .

Moreover, if R is an $N \times N$ system of properly supported classical pseudodifferential operators in Y , then each term in the asymptotic expansion of the symbol of R has vanishing Taylor coefficients at a point $(y, \eta) \in K'$ if and only if each term in the asymptotic expansion of the symbol of the pseudodifferential operator ARB in X has vanishing Taylor coefficients at $\chi(y, \eta) \in K$.

Proof. Let $P_{11} = (Q_{jk})$ and choose any properly supported scalar Fourier integral operators $A \in I_{\text{cl}}^{1-m}(X \times Y, \Gamma')$ and $B \in I_{\text{cl}}^0(Y \times X, (\Gamma^{-1})')$ such that the principal symbol of BA is equal to q in a conic neighborhood Ω of K' . Since $q \neq 0$ we find that A and B are non-characteristic at the restriction of the graphs of χ and χ^{-1} to K' and to K respectively. We may choose A and B such that $WF'(A)$ and $WF'(B)$ are contained in small conic neighborhoods. Since $\sigma(P_{11}) = \lambda \text{Id}_J$ it follows that the principal symbol of $AQ_{kk}B$ is equal to $\xi_1 + if(x, \xi')$ in a neighborhood of K for $1 \leq k \leq J$.

Now choose $A' \in I_{\text{cl}}^0(X \times Y, \Gamma')$ and $B' \in I_{\text{cl}}^{m-1}(Y \times X, (\Gamma^{-1})')$ properly supported and such that

$$\begin{aligned} K \cap WF(AB' - \text{Id}) &= \emptyset, & K' \cap WF(B'A - \text{Id}) &= \emptyset, \\ K \cap WF(A'B - \text{Id}) &= \emptyset, & K' \cap WF(BA' - \text{Id}) &= \emptyset. \end{aligned}$$

Naturally, these conditions continue to hold with Id replaced by Id_N if A is replaced by $A\text{Id}_N$, and A' , B and B' are similarly replaced by $N \times N$ systems. The systems $\tilde{A} = A\text{Id}_N$ and $\tilde{B} = B\text{Id}_N$ thus constructed satisfy (i), and it is also clear that (ii) holds. Moreover, if $R = (R_{jk})$ is an $N \times N$ system of properly supported classical pseudodifferential operators in Y such that each term in the asymptotic expansion of the symbol of R has vanishing Taylor coefficients at a point $(y, \eta) \in K'$, then each term in the asymptotic expansion of the symbol of $\tilde{A}R\tilde{B} = (AR_{jk}B)$ has vanishing Taylor coefficients at the point $\chi(y, \eta) \in K$ by [16, Lemma A.1] applied to $AR_{jk}B$ for $j, k = 1, \dots, N$. Conversely, if each term in the asymptotic expansion of the symbol of $\tilde{A}R\tilde{B}$ has vanishing Taylor coefficients at a point $\chi(y, \eta) \in K$, then the same argument shows that each term in the asymptotic expansion of the symbol of $\tilde{B}'\tilde{A}R\tilde{B}\tilde{A}'$ has vanishing Taylor coefficients at the point $(y, \eta) \in K'$, where $\tilde{A}' = A'\text{Id}_N$ and $\tilde{B}' = B'\text{Id}_N$. Since $\tilde{B}'\tilde{A}R\tilde{B}\tilde{A}' \equiv R \pmod{\Psi^{-\infty}}$ near K' , this completes the proof. \square

Let $\{e_k : k = 1, \dots, n\}$ be a basis for \mathbb{R}^n , let (U, x) be local coordinates on a smooth manifold X of dimension n , and let

$$\left\{ \frac{\partial}{\partial x_k} : k = 1, \dots, n \right\}$$

be the induced local frame for the tangent bundle TX . For a matrix valued function $f \in C^\infty(U, \mathcal{L}_N)$ we can use standard multi-index notation to express the partial derivatives of f since the local frame fields commute. If $\alpha \in \mathbb{N}^n$ is a multi-index we shall by $\partial_x^\alpha f(\gamma)$ denote the matrix $(\partial_x^\alpha f_{ij}(\gamma))$ if $f(\gamma) = (f_{ij}(\gamma))$.

Lemma A.2. *Let X be a smooth manifold of dimension n , and for $j \geq 1$ let $p, q_j, g_j \in C^\infty(X)$ be $N \times N$ systems. Let $\{\gamma_j\}_{j=1}^\infty$ be a sequence in X such that $\gamma_j \rightarrow \gamma$ as $j \rightarrow \infty$, and assume that $p(\gamma) = p(\gamma_j) = 0$ for all j . Assume also that p is of principal type at γ , that is, there exists a tangent vector $\partial_\nu \in T_\gamma X$ such that*

$$\partial_\nu p(\gamma) : \text{Ker } p(\gamma) \longrightarrow \text{Coker } p(\gamma) = \mathbb{C}^N / \text{Ran } p(\gamma)$$

is bijective. Let (U, x) be local coordinates on X near γ , and suppose that there exists an $N \times N$ system $q \in C^\infty(X)$ such that

$$\partial_x^\alpha q(\gamma) = \lim_{j \rightarrow \infty} \partial_x^\alpha q_j(\gamma_j)$$

for all $\alpha \in \mathbb{N}^n$. If $q_j - pg_j$ vanishes of infinite order at γ_j for all j , then there exists an $N \times N$ system $g \in C^\infty(X)$ such that $q - pg$ vanishes of infinite order at

γ . Furthermore,

$$(A.2) \quad \partial_x^\alpha g(\gamma) = \lim_{j \rightarrow \infty} \partial_x^\alpha g_j(\gamma_j)$$

for all $\alpha \in \mathbb{N}^n$.

Note that in view of Borel's theorem, the assumption concerning the existence of q is equivalent to assuming that all the limits $\lim_{j \rightarrow \infty} \partial_x^\alpha q_j(\gamma_j)$ exist.

Proof. First note that although the result is stated for a manifold, it is purely local so we may assume that $X \subset \mathbb{R}^n$ in the proof. Next we observe that $p(\gamma) = 0$ implies that $\text{Ker } p(\gamma) = \mathbb{C}^N = \text{Coker } p(\gamma)$. Thus $\partial_\nu p(\gamma)$ is invertible, so $|\partial_\nu p(\gamma)| \neq 0$ which means we can find a neighborhood \mathcal{U} of γ where $|\partial_\nu p(\gamma)| \neq 0$. Hence the matrix valued function $\partial_\nu p(w)$ is invertible in \mathcal{U} , and we let $(\partial_\nu p(w))^{-1}$ denote its inverse. By Cramer's rule it follows that $(\partial_\nu p)^{-1}$ is C^∞ in \mathcal{U} . We may without loss of generality assume that $\gamma_j \in \mathcal{U}$ for $j \geq 1$.

Moreover, we have that $\partial_{\lambda\nu} p(\gamma) = \lambda \partial_\nu p(\gamma)$ is invertible for any $0 \neq \lambda \in \mathbb{R}$ so we may assume that ν as a vector in \mathbb{R}^n has length 1. (We will identify a tangent vector $\nu \in \mathbb{R}^n$ at γ with $\partial_\nu \in T_\gamma \mathbb{R}^n$ through the usual vector space isomorphism.) By an orthonormal change of coordinates we may even assume that $\partial_\nu p(w) = \partial_{e_1} p(w)$. In accordance with the notation used in the statement of the lemma, we shall write $\partial_{x_k} p(w)$ for the partial derivatives $\partial_{e_k} p(w)$ and denote by $(\partial_{x_1} p(w))^{-1}$ the inverse of $\partial_{x_1} p(w) = \partial_{x_1} p(w)$ in \mathcal{U} . Now

$$(A.3) \quad 0 = \partial_{x_1}(q_j - pg_j)(\gamma_j) = \partial_{x_1} q_j(\gamma_j) - \partial_{x_1} p(\gamma_j) g_j(\gamma_j)$$

for all j since $p(\gamma_j) = 0$. Since $\lim_j \partial_{x_1} q_j(\gamma_j) = \partial_{x_1} q(\gamma)$ by assumption, equation (A.3) yields

$$\lim_{j \rightarrow \infty} g_j(\gamma_j) = (\partial_{x_1} p(\gamma))^{-1} \partial_{x_1} q(\gamma) = a \in \mathcal{L}_N,$$

and we claim that we can in the same way determine

$$\lim_{j \rightarrow \infty} \partial_x^\alpha g_j(\gamma_j) = a_{(\alpha)} \in \mathcal{L}_N$$

for any $\alpha \in \mathbb{N}^n$. In fact, arguing by contradiction, we introduce a total well-ordering of the derivatives ∂_x^α by means of a monomial ordering of the corresponding monomials x^α . We choose the graded reverse lexicographic order $>_{\text{grevlex}}$ together with the (non-conventional) ordering $x_n > \dots > x_1$ of the variables. That is to say, to determine if $\partial_x^\alpha >_{\text{grevlex}} \partial_x^\beta$ for multi-indices $\alpha, \beta \in \mathbb{N}^n$, we first compare the total lengths $|\alpha|$ and $|\beta|$, and in case of equality compare the left-most entries α_1 and β_1 , but reversing the outcome so that the multi-index with the smaller entry yields a larger derivative in the ordering. In case of a tie this is followed by a similar comparison of the second entries from the left and so forth, ending with a comparison of the right-most entries. This will then lead to the ordering $x_n > \dots > x_1$ of the variables, in the sense that the tangent vectors are ordered $\partial_{x_n} >_{\text{grevlex}} \dots >_{\text{grevlex}} \partial_{x_1}$. Suppose now that ∂_x^α is the first derivative such that the limit of $\partial_x^\alpha g_j(\gamma_j)$ does not exist as $j \rightarrow \infty$. Let ε_k be the k :th basis vector in \mathbb{R}^n , and consider the limit of $\partial_x^{\alpha+\varepsilon_1}(q_j - pg_j)(\gamma_j)$ as $j \rightarrow \infty$. By Leibniz's formula we have

$$\partial_x^{\alpha+\varepsilon_1}(q_j - pg_j) = \partial_x^{\alpha+\varepsilon_1} q_j - p \partial_x^{\alpha+\varepsilon_1} g_j - \partial_{x_1} p \partial_x^\alpha g_j - \sum_{\{\beta: \beta < \alpha\}} \binom{\alpha}{\beta} \partial_{x_1} (\partial_x^{\alpha-\beta} p \partial_x^\beta g_j).$$

Note that if $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\alpha_1 \geq 1$, then the sum over β in the right-hand side contains an additional term of the form $\alpha_1 \partial_{x_1} p \partial_x^\alpha g_j$, produced by the value $\beta = \alpha - \varepsilon_1$. Evaluating at γ_j we find that the left-hand side converges to 0 as $j \rightarrow \infty$ by assumption, and since $p(\gamma_j) = 0$ it follows from our choice of ordering that with the exception of the term $(\alpha_1 + 1) \partial_{x_1} p \partial_x^\alpha g_j$, all other terms have well-defined limits as $j \rightarrow \infty$. Arguing as in the discussion following (A.3), we can therefore determine the limit of $\partial_x^\alpha g_j(\gamma_j)$ as $j \rightarrow \infty$ by multiplying with $(\alpha_1 + 1)^{-1} (\partial_{x_1} p(\gamma_j))^{-1}$ from the left. This contradiction proves the claim.

By using Borel's theorem for each entry it is clear that there exists a matrix valued function $g \in C^\infty(X, \mathcal{L}_N)$ such that

$$\partial_x^\alpha g(\gamma) = a_{(\alpha)} = \lim_{j \rightarrow \infty} \partial_x^\alpha g_j(\gamma_j)$$

for all $\alpha \in \mathbb{N}^n$. Since $q - pg$ vanishes of infinite order at γ by construction, this completes the proof. \square

Keeping the notation from Lemma A.2, there is naturally an analogue result if p is an elliptic system. In fact, very little has to be changed for the proof to work in this setting: we essentially just replace $w \mapsto (\partial_\nu p(w))^{-1}$ with the inverse $w \mapsto p(w)^{-1}$ of p . The ordering used in the proof can be the same; the only feature needed in this case is that it is a graded ordering. The result is stated below for easy reference. We omit the proof.

Lemma A.3. *Let X be a smooth manifold of dimension n , and for $j \geq 1$ let $p, q_j, g_j \in C^\infty(X)$ be $N \times N$ systems. Let $\{\gamma_j\}_{j=1}^\infty$ be a sequence in X such that $\gamma_j \rightarrow \gamma$ as $j \rightarrow \infty$, and assume that $|p(\gamma)|$ and $|p(\gamma_j)|$ are non-vanishing for all j , where $|p|$ is the determinant of p . Let (U, x) be local coordinates on X near γ , and suppose that there exists an $N \times N$ system $q \in C^\infty(X)$ such that*

$$\partial_x^\alpha q(\gamma) = \lim_{j \rightarrow \infty} \partial_x^\alpha q_j(\gamma_j)$$

for all $\alpha \in \mathbb{N}^n$. If $q_j - pg_j$ vanishes of infinite order at γ_j for all j , then there exists an $N \times N$ system $g \in C^\infty(X)$ such that $q - pg$ vanishes of infinite order at γ . Furthermore,

$$\partial_x^\alpha g(\gamma) = \lim_{j \rightarrow \infty} \partial_x^\alpha g_j(\gamma_j)$$

for all $\alpha \in \mathbb{N}^n$.

The method used in the proof of Lemma A.2 can also be applied to prove the following result for certain functions of non-principal type. We only need the result for scalar functions but combined with the first part of the proof of Lemma A.2, the proof would work equally well for systems.

Lemma A.4. *Let X be a smooth manifold of dimension n , and let λ and e be scalar functions in $C^\infty(X)$. Let $\gamma \in X$ and assume that $\lambda(\gamma) = 0$ and $d\lambda(\gamma) \neq 0$. If $\lambda^m e$ vanishes of infinite order at γ for some $m \geq 1$, then e vanishes of infinite order at γ .*

Proof. As in the proof of Lemma A.2 we conclude that since the statement is local we may assume that $X \subset \mathbb{R}^n$ and $\partial\lambda(\gamma)/\partial x_1 \neq 0$, where we use coordinates x_1, \dots, x_n in X . Let $>_{\text{greylex}}$ be the total well-ordering of the derivatives ∂_x^α introduced in the proof of Lemma A.2, and assume that $\partial_x^\alpha e(\gamma)$ is the first derivate of e

that does not vanish at γ . Let ε_j be the j :th basis vector in \mathbb{R}^n , and consider the derivative $\partial_{x_1}^m \partial_x^\alpha (\lambda^m e)(\gamma)$. By Leibniz's formula we have

$$\begin{aligned} \partial_{x_1}^m \partial_x^\alpha (\lambda^m e) &= \sum_{k=0}^m \binom{m}{k} \partial_{x_1}^k (\lambda^m) \partial_x^{\alpha + (m-k)\varepsilon_1} e \\ &\quad + \sum_{k=0}^m \binom{m}{k} \sum_{\{\beta: \beta < \alpha\}} \binom{\alpha}{\beta} \partial_x^{\alpha - \beta + k\varepsilon_1} (\lambda^m) \partial_x^{\beta + (m-k)\varepsilon_1} e. \end{aligned}$$

In the first sum, all terms with $k < m$ vanish at γ since $\lambda(\gamma) = 0$, so the only contribution we get is $m!(\partial\lambda(\gamma)/\partial x_1)^m \partial_x^\alpha e(\gamma)$. This also implies that all terms in the double sum with $|\beta| + m - k > |\alpha|$ vanish at γ . Conversely, if $|\beta| + m - k < |\alpha|$ then $\partial_x^{\beta + (m-k)\varepsilon_1} e(\gamma) = 0$ since the ordering is graded. When we have equality $|\beta| + m - k = |\alpha|$ in the double sum then $m - k \geq 1$ since $\beta < \alpha$, so we can write

$$(A.4) \quad \beta + (m - k)\varepsilon_1 = \alpha - \sum_{\ell=1}^{m-k} \varepsilon_{j_\ell} + (m - k)\varepsilon_1$$

for some ε_{j_ℓ} with $1 \leq j_\ell \leq n$. Unless the left-most entry α_1 of α is $\geq m - k$ so that we can choose $\varepsilon_{j_\ell} = \varepsilon_1$ for all $1 \leq j_\ell \leq n$ in (A.4), we thus have $\partial_x^\alpha >_{\text{grevlex}} \partial_x^{\beta + (m-k)\varepsilon_1}$ which by our assumptions implies that $\partial_x^{\beta + (m-k)\varepsilon_1} e(\gamma) = 0$. On the other hand, if $\beta + (m - k)\varepsilon_1 = \alpha$ then $\partial_x^{\alpha - \beta + k\varepsilon_1} (\lambda^m) = \partial_{x_1}^m (\lambda^m)$ so this produces another term of the form $m!(\partial\lambda(\gamma)/\partial x_1)^m \partial_x^\alpha e(\gamma)$. Hence

$$0 = \partial_{x_1}^m \partial_x^\alpha (\lambda^m e)|_\gamma = C(\partial\lambda(\gamma)/\partial x_1)^m \partial_x^\alpha e(\gamma)$$

where C is a positive constant depending only on m and α . Thus the right-hand side is non-vanishing by our assumptions, and this contradiction proves the claim. \square

Lemma A.2 will be used to prove the following result for homogeneous systems on the cotangent bundle. First, recall that if M is the map given by (2.2), then the radial vector field $\rho \in T(T^*(X) \setminus 0)$ is defined by

$$\rho f = \frac{d}{dt} M_t^* f|_{t=1}, \quad f \in C^\infty(T^*(X) \setminus 0).$$

In terms of local coordinates we have $\rho(w) = \xi \partial_\xi$ if $w = (x, \xi)$, see the discussion following [10, Definition 21.1.8]. Moreover, if f is homogeneous of degree ℓ , then differentiation gives $\rho f = \ell f$ by Euler's homogeneity relation.

Proposition A.5. *For $j \geq 1$ let $p, q_j, g_j \in C^\infty(T^*(\mathbb{R}^n) \setminus 0)$ be $N \times N$ systems, where p and q_j are homogeneous of degree m and g_j is homogeneous of degree 0. Let $\{\gamma_j\}_{j=1}^\infty$ be a sequence in $T^*(\mathbb{R}^n) \setminus 0$ such that $\gamma_j \rightarrow \gamma$ as $j \rightarrow \infty$, and assume that $p(\gamma) = p(\gamma_j) = 0$ for all j . Assume also that p is of principal type at γ , that is, there exists a tangent vector $\partial_\nu \in T_\gamma T^*(\mathbb{R}^n)$ such that*

$$\partial_\nu p(\gamma) : \text{Ker } p(\gamma) \longrightarrow \text{Coker } p(\gamma) = \mathbb{C}^N / \text{Ran } p(\gamma)$$

is bijective. If there exists an $N \times N$ system $q \in C^\infty(T^(\mathbb{R}^n) \setminus 0)$, homogeneous of degree m , such that*

$$\partial_x^\alpha \partial_\xi^\beta q(\gamma) = \lim_{j \rightarrow \infty} \partial_x^\alpha \partial_\xi^\beta q_j(\gamma_j)$$

for all $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n$, and if $q_j - pg_j$ vanishes of infinite order at γ_j for all j , then there exists an $N \times N$ system $g \in C^\infty(T^*(\mathbb{R}^n) \setminus 0)$, homogeneous of degree 0, such that $q - pg$ vanishes of infinite order at γ . Furthermore,

$$(A.5) \quad \partial_x^\alpha \partial_\xi^\beta g(\gamma) = \lim_{j \rightarrow \infty} \partial_x^\alpha \partial_\xi^\beta g_j(\gamma_j)$$

for all $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n$.

Proof. Let $\pi : T^*(\mathbb{R}^n) \setminus 0 \rightarrow S^*(\mathbb{R}^n)$ be the projection, and identify $S^*(\mathbb{R}^n)$ with $\mathbb{R}^n \times S^{n-1}$. We have $\text{Ker } p(\gamma) = \mathbb{C}^N = \text{Coker } p(\gamma)$, so $\partial_\nu p(\gamma)$ is invertible. It follows that $\partial_{\lambda\nu} p(\gamma) = \lambda \partial_\nu p(\gamma)$ is invertible for all $\lambda > 0$. With $\gamma = (x_0, \xi_0)$ and $\nu = (\nu_1, \nu_2)$ this implies that $\partial_\mu p(\pi(\gamma))$ is invertible for $\mu = (\nu_1, \nu_2/|\xi_0|)$ since

$$\begin{aligned} \partial_\nu p(\gamma) &= \frac{d}{dt} p(\gamma + t\nu)|_{t=0} \\ &= \frac{d}{dt} (|\xi_0|^m p(x_0 + t\nu_1, (\xi_0 + t\nu_2)/|\xi_0|))|_{t=0} = |\xi_0|^m \partial_\mu p(\pi(\gamma)). \end{aligned}$$

By using the homogeneity of p , q , q_j and g_j we may then assume that γ and γ_j belong to $S^*(\mathbb{R}^n)$ for $j \geq 1$ to begin with, and that $\partial_\nu p(\gamma)$ is invertible with ν replaced by μ .

We may also assume that ν is a tangent vector $\nu \in T_\gamma S^*(\mathbb{R}^n)$. Indeed, the radial vector field ρ applied k times to $a \in C^\infty(T^*(\mathbb{R}^n) \setminus 0)$ equals $\ell^k a$ if a is homogeneous of degree ℓ . For any point $w \in S^*(\mathbb{R}^n)$ with $w = (w_x, w_\xi)$ in local coordinates on $T^*(\mathbb{R}^n)$ it is easy to see that

$$T_w S^*(\mathbb{R}^n) = \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^n : \langle w_\xi, v \rangle = 0\}.$$

Therefore a basis for $T_w S^*(\mathbb{R}^n)$ together with the radial vector field $\rho(w)$ at w constitutes a basis for $T_w T^*(\mathbb{R}^n)$. By these considerations it follows that ∂_ν cannot be a multiple of the radial vector field at γ since $\partial_\nu p(\gamma)$ is invertible while $p(\gamma) = 0$. Hence, $\partial_\nu = c\rho(\gamma) + \partial_{\tilde{\nu}}$ for some $c \in \mathbb{R}$ and $0 \neq \tilde{\nu} \in T_\gamma S^*(\mathbb{R}^n)$. Again, since $p(\gamma) = 0$ we have $\partial_{\tilde{\nu}} p(\gamma) = \partial_\nu p(\gamma)$ by Euler's homogeneity relation, which proves the claim. Note that these arguments also show that if we can find a homogeneous matrix valued function g such that $q - pg$ vanishes of infinite order in the directions $T_\gamma S^*(\mathbb{R}^n)$, then $q - pg$ vanishes of infinite order at γ , for the derivatives involving the radial direction are determined by lower order derivatives in the directions $T_\gamma S^*(\mathbb{R}^n)$.

Write $p(x, \xi) = |\xi|^m \pi^* p_s(x, \xi)$, where $p_s = p \circ \pi$ is the restriction of p to $S^*(\mathbb{R}^n)$. Doing the same for q , q_j and g_j we find by the hypotheses of the proposition together with an application of Lemma A.2, that there exists a matrix valued function $g_s \in C^\infty(S^*(\mathbb{R}^n), \mathcal{L}_N)$, such that $q_s - p_s g_s$ vanishes of infinite order at γ and (A.2) holds for g_s , interpreted in the appropriate sense for a local frame for $S^*(\mathbb{R}^n)$. The function $g(x, \xi) = \pi^* g_s(x, \xi)$ is homogeneous of degree 0 and coincides with g_s on $S^*(\mathbb{R}^n)$. In particular, all derivatives of g and g_s in the directions $T_\gamma S^*(\mathbb{R}^n)$ are equal at γ . Thus, by the arguments above we conclude that $q - pg$ vanishes of infinite order at γ . Since g and g_j are homogeneous of degree 0, the same arguments also imply that (A.5) holds for g , which completes the proof. \square

Using Lemma A.3 in place of Lemma A.2 we obtain the following result for elliptic systems corresponding to Proposition A.5.

Proposition A.6. *For $j \geq 1$ let $p, q_j, g_j \in C^\infty(T^*(\mathbb{R}^n) \setminus 0)$ be $N \times N$ systems, where p and q_j are homogeneous of degree m and g_j is homogeneous of degree 0.*

Let $\{\gamma_j\}_{j=1}^\infty$ be a sequence in $T^*(\mathbb{R}^n) \setminus 0$ such that $\gamma_j \rightarrow \gamma$ as $j \rightarrow \infty$, and assume that $|p(\gamma)|$ and $|p(\gamma_j)|$ are non-vanishing for all j , where $|p|$ is the determinant of p . If there exists an $N \times N$ system $q \in C^\infty(T^*(\mathbb{R}^n) \setminus 0)$, homogeneous of degree m , such that

$$\partial_x^\alpha \partial_\xi^\beta q(\gamma) = \lim_{j \rightarrow \infty} \partial_x^\alpha \partial_\xi^\beta q_j(\gamma_j)$$

for all $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n$, and if $q_j - pg_j$ vanishes of infinite order at γ_j for all j , then there exists an $N \times N$ system $g \in C^\infty(T^*(\mathbb{R}^n) \setminus 0)$, homogeneous of degree 0, such that $q - pg$ vanishes of infinite order at γ . Furthermore,

$$\partial_x^\alpha \partial_\xi^\beta g(\gamma) = \lim_{j \rightarrow \infty} \partial_x^\alpha \partial_\xi^\beta g_j(\gamma_j)$$

for all $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n$.

Proof. Let $\pi : T^*(\mathbb{R}^n) \rightarrow S^*(\mathbb{R}^n)$ be the projection, and identify $S^*(\mathbb{R}^n)$ with $\mathbb{R}^n \times S^{n-1}$. Arguing as in the proof of Proposition A.5, it follows by homogeneity that all assumptions continue to hold after projecting onto the cosphere bundle. An application of Lemma A.3 yields the existence of a matrix valued function $g_s \in C^\infty(S^*(\mathbb{R}^n), \mathcal{L}_N)$ for which the pullback $g = \pi^*g_s$ has the required properties. This completes the proof. \square

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